BALANCED SUBGROUPS

OF

ABELIAN GROUPS

by

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A thesis presented to the

Australian National University

for the degree of

Doctor of Philosophy

Canberra

March, 1975

STATEMENT

The work in this thesis is my own unless otherwise indicated.

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ACKNOWLEDGEMENTS

My warmest thanks go to Dr K.M. Rangaswamy who supervised this work. I am grateful to him for his guidance, numerous suggestions and constant encouragement.

I also thank my co-supervisor Dr L.G. Kovács for his most helpful suggestions.

My thanks go also to my wife Sally for her voluminous patience during the past year.

I am grateful for the award of a Commonwealth Scholarship and to the Department of Mathematics, Institute of Advanced Studies, of the Australian National University for having me as a Research Student.

Finally, I thank Mrs Barbara Geary for an excellent typing job.

ABSTRACT

In 1967, Hill extended Ulm's classification theorem for reduced countable p-groups to the class of totally projective Abelian p-groups introduced by Nunke. Hill has also shown that the class of totally projective Abelian p-groups is the class of projectives relative to a class of short exact sequences (called balanced by Fuchs) of p-groups.

In this thesis, balanced sequences of arbitrary Abelian groups are defined and the corresponding classes of projectives and injectives studied. The properties of balanced sequences are explored and it is shown that the new definition specialises, in the case of torsion groups, to that of Fuchs. The class of injectives relative to balanced sequences is shown to coincide with the well known class of pure injectives.

In order to study the balanced projectives, a theory of mixed Abelian groups of torsion free rank 1 is developed, considering them as extensions of a cyclic group by a torsion group. It is shown that a balanced projective is a direct summand of a direct sum of groups having torsion free rank at most 1, where each group of torsion free rank at most 1 involved is an extension of a cyclic group by a totally projective torsion group.

Certain closure properties enjoyed by the class of totally projective p-groups are shown to fail for the class of balanced projectives. However, it is shown that the class of projectives relative to a subclass of the class of balanced sequences has the required closure properties and that this new class is only slightly larger than the class of balanced projectives.

The thesis concludes with an investigation of a number of questions of the following type. Given a class X of Abelian groups (such as the class of all torsion groups), for what Abelian groups Y is it true that all balanced extensions of groups in X by Y (or of Y by groups in X) must split?

CONTENTS

CHAPTER 1

INTRODUCTION

All groups under discussion will be Abelian, so the word 'group' will always designate an Abelian group.

In 1933, Ulm [1] achieved the complete classification of countable torsion groups using matrix-theoretical methods - 'undoubtedly the most striking result yet obtained in Abelian groups' (Kaplansky [1], p. 26). Briefly, Ulm's theorem says that every countable p-group is completely determined by a set of well-defined invariants, now known as the Ulm invariants. Zippin [1] gave a group theoretical proof, while Kaplansky and Mackey [1] extended the result to modules over a complete discrete valuation ring. It was not until 1958 that Kolettis [1] used the very elegant proof technique devised by Kaplansky and Mackey to extend Ulm's theorem to include direct sums of countable p-groups.

Direct sums of reduced countable p-groups have length at most Ω, the first uncountable ordinal. Using powerful homological methods, Nunke [3] discovered a remarkable generalisation of direct sums of countable groups to a class containing p-groups of arbitrary ordinal length - these he called totally projective groups. E.A. Walker conjectured that Ulm's theorem would extend to totally projective groups and in 1967 Parker and Walker [1] proved such an extension for totally projective p-groups of lengths less than $\Omega\omega$. That same year, Hill [1] announced Ulm's theorem for totally projective p-groups; the proof given by Griffith [3] is said to follow closely that given in the (as yet unpublished) paper of Hill [1].

A number of remarkable characterisations of totally projective p-groups are now known. The class of simply presented p-groups studied by Crawley and Hales [1, 2] coincides with the class of totally projective p-groups. Hill also showed that the totally projective p-groups are the projectives with respect to a class of short exact sequences of p-groups, called balanced-exact sequences by Fuchs [2]. Parker and Walker [1] correctly conjectured that the class of totally projective p-groups is the only class P of p-groups whose members are determined up to isomorphism by their Ulm invariants and such that

(a) P contains the cyclic group of order p,

(b) P is closed under the operations of forming arbitrary direct sums and direct summands, and

(c) if A is a group and σ is an ordinal then $A \in P$ if and only if $p^{\sigma}A \in P$ and $A/p^{\sigma}A \in P$.

It follows that in general every larger class containing the totally projective p-groups will require additional invariants for classification of its members up to isomorphism. A natural approach to the problem of finding such a class is to generalise a particular property of the totally projective p-groups, and study the class of groups with this new property. For example Warfield [1] has obtained significant extensions of Ulm's theorem to simply presented modules over a discrete valuation ring.

In this thesis, the projective property of totally projective p-groups is singled out for generalisation; we define balanced (exact) sequences of arbitrary groups and investigate the corresponding classes of projectives and injectives.

After detailing a number of standard concepts, Chapter 2 deals with properties of the height matrix, a device introduced by Rotman as a natural generalisation of height. With each height matrix $\mathbb{M}$ and group A we associate a functorial subgroup $A(\mathbb{M})$ which generalises the subgroups of the form $p^{\sigma}A$. An 'addition' is defined for height matrices; it is shown that if $\mathbb{M}$ and $\mathbb{N}$ are height matrices and A is a group then $[A(\mathbb{M})](\mathbb{N}) = A(\mathbb{M} + \mathbb{N})$. The chapter concludes with a brief survey of results on totally projective groups, together with some recent generalisations useful in later work.

In Chapter 3, the height matrix is used to generalise balanced subgroups and balanced sequences to arbitrary groups. The salient properties of the corresponding balanced subgroups are studied and a number of alternate characterisations given. We show that the class of balanced sequences forms a proper class in the sense of Mac Lane [1]. A class of subgroups studied by Warfield [1, 2] and Wick [1] (here called H-balanced subgroups) is also discussed.

In Chapter 4 we show that the balanced injectives are just the algebraically compact groups of Kaplansky [1]. These groups are well known and are in fact characterised by isomorphism invariants. Our argument in (4.3) extends that given by Griffith [2], who determined the balanced injectives in the category of all p-groups. A consequence of this more general argument is that the injectives with respect to balanced sequences of torsion free groups are seen to be the cotorsion groups of Harrison [1]. We also answer a question of C.L. Walker [2].

In Chapter 5 we study two classes of groups (denoted by A and C) whose members have torsion free rank at most 1. Groups in A are the 'building blocks' for our study of the balanced projectives, and groups in C are similarly fundamental to our work with H-projectives. As might be expected, C is a subclass of A and each class contains the totally projective groups. Many of the results that are required for groups in these two classes also apply to arbitrary groups with torsion free rank 1. Accordingly, we first examine a general theory of groups with torsion free rank 1. Our approach is to consider such groups as extensions; if A has torsion free rank 1 and the element a in A has infinite order, then A is an extension

$$0 \to \langle a \rangle \to A \to T^* \to 0 \tag{1}$$

of the free group $\langle a \rangle$ by the torsion group T^*.

With each extension (1) we associate the pair (M, T^*) where M is the height matrix of a. An equivalence relation is then defined on the class of all pairs $(\mathfrak{M}, T^*)$, using only $\mathfrak{M}$ and the Ulm invariants of T^* . The class A is defined to be made up of the totally projective groups, together with all groups A which can be represented as extensions (1) with T^* totally projective. We then show that any A in A , when written as an extension (1), is determined up to isomorphism by $\mathfrak{M}$ and the Ulm invariants of T^* and that two equivalent pairs associated with groups in A determine isomorphic groups in A .

On the other hand, necessary conditions (in terms of $\mathfrak{M}$ and the Ulm invariants of T^*) are given for an arbitrary pair $(\mathfrak{M}, T^*)$ to be associated with some exact sequence (1) and these conditions are shown to be sufficient when T^* is totally projective, that is, when $A \in \mathrm{A}$.

It is shown that the torsion part of a group in A is an S-group in the sense of Warfield [1]. The groups in A also allow us to show that with each height matrix, there is associated a cotorsion functor in the sense of Nunke [1] and we determine exactly when these functors have 'enough' projectives.

In Chapter 6 we characterise the balanced projectives in the category of all Abelian groups: they are just the direct summands of direct sums of members of the class A . The properties of balanced projectives are explored. Every group is the balanced image of a balanced projective. If A is balanced projective, then $A/T(A)$ is completely decomposable and $T(A)$ is a summand of an S-group ($T(A)$ denotes the torsion part of A). Thus a torsion (torsion free) group is balanced projective in the category of all Abelian groups if and only if it is totally projective (completely decomposable).

It turns out that property (c) of p. 2 does not generalise in the obvious way to our situation; we give an example of a balanced projective group A and prime p for which $p^{\omega}A$ is not balanced projective. This leads us to consider the smallest class P of groups which contains all balanced projectives, has property (b) of p. 2, and is such that if $\mathfrak{M}$ is a height matrix and $A \in P$ then $A(\mathfrak{M}) \in P$ and $A/A(\mathfrak{M}) \in P$. This turns out to be precisely the class of projectives with respect to certain balanced sequences which we call C-balanced. The essential difference between a balanced sequence and a C-balanced sequence is shown to occur with the torsion and this is used to show that when an exact sequence $0 \to B \to A \to C \to 0$ is C-balanced then so is $0 \to B(\mathfrak{M}) \to A(\mathfrak{M}) \to C(\mathfrak{M}) \to 0$ for every height matrix $\mathfrak{M}$.

The remaining two chapters contain a discussion of some problems and some partial results that can now be derived with relatively little effort. They are included here for that reason rather than as rivals for the main chapters.

In Chapter 7, invariants for the various classes of projectives are studied. We give invariants and a classification theorem for direct sums of groups in C using a method similar to that given in Fuchs [2] for the classification of completely decomposable torsion free groups. A result of Warfield [4] is given which shows that every H-projective is such a direct sum, thus yielding a complete classification of the H-projectives by invariants. The problem of finding invariants for balanced projectives is then discussed. An invariant is introduced for groups having a decomposition basis in the sense of Rotman [1].

In Chapter 8 we explore the properties of $\mathrm{Bext}(C, A)$ for groups A and C - that is, the group of balanced extensions of A by C . We introduce the functor Bext which is naturally associated with $\mathrm{Bext}(C, A)$ and indicate some of its homological properties. Various classes of groups for which $\mathrm{Bext}(C, A) = 0$ are then characterised. In particular, $\mathrm{Bext}(C, A) = 0$ for all torsion free groups C if and only if A is cotorsion. On the other hand, the groups C for which $\mathrm{Bext}(C, A) = 0$ for all torsion free groups A are characterised as homomorphic images of direct sums of groups in C where the homomorphisms have torsion kernel, or equivalently as groups C for which $C/T(C)$ is completely decomposable. It turns out that every group with a decomposition basis, and every summand of a direct sum of mixed groups with torsion free rank 1 has this property. We show that a group A satisfies $\mathrm{Bext}(C, A) = 0$ for all torsion groups C exactly when $T(A)$ is torsion complete. We conclude with some results on groups which can be represented as the balanced extension of a torsion group by a torsion free group - that is, groups with balanced torsion part.

CHAPTER 2

PRELIMINARIES

Basic notation, definitions and results

Throughout this thesis the symbol A will be reserved for an Abelian group and $T(A)$ will denote the torsion part of A ; if p is a prime then A_p will denote the p-component of A , and $A[p]$ the subgroup $\{a \in A : pa = 0\}$. The standard notation Q, Z, $Z(p^\infty)$, $Z(p^n)$ will be used for the group of rationals, the group of integers, the quasi-cyclic p-group and the cyclic group of order p^n , respectively. The *torsion free rank* of A is the Q-rank of $Q \otimes A$. Denote by Z_p the ring of integers localised at the prime p , that is, the subring of Q consisting of those rationals whose denominators are prime to p . Every Z_p-module can be regarded as a group which is uniquely divisible by all primes except p , and results for Z_p-modules will on occasion be assumed from corresponding results for groups. The group $\mathrm{Ext}(Q/Z_p, A)$ will be denoted by $c_p(A)$, and $\mathrm{Ext}(Q/Z, A)$ will be denoted by $c(A)$.

For each prime p and ordinal σ the subgroup $p^\sigma A$ of A is defined inductively as follows: starting with $p^0 A = A$ we set $p^\sigma A = p(p^\rho A)$ when $\sigma = \rho + 1$, and $p^\sigma A = \bigcap_{\rho<\sigma} p^\rho A$ when σ is a limit ordinal. The *maximal p-divisible subgroup* of A is denoted by $p^\infty A$, and is $p^\sigma A$ for any ordinal σ such that $p^\sigma A = p^{\sigma+1} A$. The least ordinal σ such that $p^\sigma A = p^\infty A$ is called the *reduced p-length* of A and is denoted by $l_p(A)$. Since $p^\sigma A \leq p^\infty A$ for every ordinal σ then with the informality customary in this field we shall always write $\sigma \leq \infty$. We also observe the conventions

$\infty + \sigma = \infty = \sigma + \infty$ for each ordinal σ and $\infty - 1 = \infty$, and say that $\infty - 1$ exists just as we say $\sigma - 1$ exists when $\sigma > 0$ and σ is not a limit ordinal. The *p-height* of an element a in A will be denoted by $h_p^A(a)$ (or just $h_p(a)$ when the context is clear); if $a \in p^\infty A$ then $h_p(a) = \infty$, otherwise $h_p(a)$ is the (unique) ordinal σ for which $a \in p^\sigma A \setminus p^{\sigma+1}A$. Note that if p is a prime and k is a positive integer such that $(p, k) = 1$ then $h_p(a) = h_p(ka)$ for every a in A.

For convenience, the subgroup $(p^\sigma A)[p]$ of A will always be denoted by $p^\sigma A[p]$. Let B be a subgroup of A. If σ is an ordinal and p is a prime, denote the Z/pZ dimension of $p^\sigma A[p]/p^{\sigma+1}A[p]$ by $f_\sigma^p(A)$ and the Z/pZ dimension of $p^\sigma A[p]/\left((p^{\sigma+1}A+B) \cap p^\sigma A[p]\right)$ by $f_\sigma^p(A, B)$. We also write $f_\infty^p(A)$ for the Z/pZ dimension of $p^\infty A[p]$ and $f_\infty^p(A, B)$ for the Z/pZ dimension of $p^\infty A[p]/(B \cap p^\infty A[p])$. The cardinals $f_\sigma^p(A)$ are known as the *Ulm invariants* of A, and the $f_\sigma^p(A, B)$ are called the *relative Ulm invariants* of A with respect to B.

If B is a subgroup of A, then B is *p-isotype* in A if $p^\sigma B = p^\sigma A \cap B$ for all ordinals σ, and B is *isotype* in A if B is p-isotype for every prime p. The subgroup B is said to be *pure* in A if $nB = nA \cap B$ for all integers $n \geq 0$. We remark that purity of B in A is equivalent to the exactness of

$$0 \to nB \to nA \to n(A/B) \to 0$$

for all integers $n \geq 0$ - this should be compared with the definition of a balanced subgroup given in Chapter 3. Megibben has observed the following stronger version of a well-known result on purity.

2.1 LEMMA (Megibben [1]). *If B is a subgroup of A such that A/B*

has no p-torsion, then B is p-isotype in A . In particular if A/B is torsion free then B is isotype in A . □

If σ is a given ordinal, then the subgroup B of the p-group A is said to be *σ-dense* in A if $B + p^{\rho}A = A$ whenever $\rho < \sigma$. A limit ordinal λ is *cofinal with* ω exactly when there is a countable sequence of smaller ordinals σ_i such that $\sup \sigma_i = \lambda$.

The following lemma is known.

2.2 LEMMA. *Let A be a reduced p-group such that $l_p(A) = \sigma$. Then there is a subgroup B of A such that*

(a) *B is σ-dense in A ;*

(b) *B is isotype in A ; and*

(c) $0 \neq A/B \leq Z(p^{\infty})$.

Proof. Fix a non-zero element a in $A[p]$ and put $\delta = h_p(a)$. For each ordinal ρ such that $\delta \leq \rho < \sigma$ we obtain a subgroup G_ρ of $A[p]$ such that $G_\rho \oplus p^{\rho}A[p] = A[p]$, no G_ρ contains a and $G_\rho \leq G_\gamma$ whenever $\rho < \gamma < \sigma$. The G_ρ are chosen inductively as follows. Start with G_δ being any complementary summand of $p^{\delta}A[p]$ in $A[p]$. Once G_τ is defined for all ordinals τ with $\delta \leq \tau < \rho$, choose G_ρ as a complement of $p^{\rho}A[p]$ in $A[p]$ containing $\bigcup_{\delta \leq \tau < \rho} G_\tau$ but not a . (The possibility of choosing such complements follows from the fact that in a vector space the intersection of the complements of a non-zero subspace is zero.) For each ρ such that $\delta \leq \rho < \sigma$ we now choose H_ρ maximal in A with respect to $H_\rho[p] = G_\rho$ and such that $H_\rho < H_\gamma$ whenever $\rho < \gamma < \sigma$. Since each H_ρ is maximal with respect to $H_\rho \cap p^{\rho}A = 0$ then it follows from Irwin and

Walker [1] that each H_ρ is isotype in A. Setting $H = \bigcup_{\delta \le \rho < \sigma} H_\rho$ then H is isotype in A and maximality of each H_ρ ensures $A = H + p^\rho A$ whenever $\rho < \sigma$. The p-height of every element in $(A/H)[p]$ is clearly greater than or equal to ρ for each ρ such that $\rho < \sigma$, so that when σ is finite, A/H is a direct sum of cycles of order σ, and when σ is infinite, Lemma 8 of Kaplansky [1] shows that A/H is divisible. Now $a \notin H$ so $A/H \ne 0$ and any summand B/H of A/H such that $0 \ne A/B \le Z(p^\infty)$ yields a B with the required properties. □

If $B \le A$ then an element a in $A \backslash B$ is *p-proper* with respect to B if $h_p^A(a) = h_p^{A/B}(a+B)$, and B is *p-nice* in A if every coset $a + B$ contains an element which is p-proper with respect to B.

A *preradical* is a function S assigning to each A a subgroup SA such that every homomorphism $\alpha : A \to B$ carries SA into SB. A *radical* is a preradical S such that, for every A, we have $S(A/SA) = 0$. An extension

$$0 \to Z \to G \to H \to 0 \tag{2}$$

yields an exact sequence

$$\ldots \to \mathrm{Hom}(G, A) \to A \to \mathrm{Ext}(H, A) \to \ldots$$

and so defines a preradical S via

$$SA = \mathrm{Im}\big(\mathrm{Hom}(G, A) \to A\big) = \mathrm{Ker}\big(A \to \mathrm{Ext}(H, A)\big)$$

(for details, see Nunke [1]); the sequence (2) is said to *represent* S. If S has a representing sequence (2) with H torsion, then S is a *cotorsion functor*. The group A is *S-projective* if $S\mathrm{Ext}(A, X) = 0$ for every group X. A cotorsion functor S has *enough projectives* if, for each A, there is an extension

$$\mathsf{e} : 0 \to N \to M \to A \to 0$$

such that $\mathsf{e} \in S\mathrm{Ext}(A, N)$ and M is S-projective. Nunke [1], to whom these ideas are due, has shown that the cotorsion functor S has enough projectives if and only if it has a representing sequence (2) such that H is S-projective and torsion, that when S has enough projectives it is a radical, and that the functors $S_{p^\sigma} : A \mapsto p^\sigma A$ are cotorsion functors with enough projectives.

If X is a subset of A then $\langle X \rangle$ denotes the subgroup of A generated by X ; if A is torsion free then $\langle X \rangle_*$ is the unique minimal pure subgroup containing X . A *class* of groups will always be understood to contain, with each member A , all groups isomorphic to A . If $\mathcal{B}$ is a class of groups, denote the class of all direct sums of groups in $\mathcal{B}$ by $\mathcal{B}^\Sigma$, and the class of all direct summands of groups in $\mathcal{B}^\Sigma$ by $\overline{\mathcal{B}}$.

Height and the height matrix

If $a \in A$, let $h_p^A(a) = \beta_p$ for each prime p . Then we define the *height* of a in A by

$$\mathsf{H}_A(a) = \langle\langle \beta_2, \beta_3, \ldots, \beta_p, \ldots \rangle\rangle .$$

We write $\mathsf{H}(a)$ when there is no danger of confusion. By a *height* K we mean an ω-sequence $\mathsf{K} = \langle\langle \beta_2, \beta_3, \ldots, \beta_p, \ldots \rangle\rangle$ of ordinals and symbols ∞ . For each height K and prime p denote the element of K corresponding to p by K_p . If $\mathsf{L} = \langle\langle \gamma_2, \gamma_3, \ldots \rangle\rangle$ is any other height then we define $\mathsf{K} \geq \mathsf{L}$ when $\beta_p \geq \gamma_p$ for all primes p . Throughout, the set of all primes will be denoted by P . For each height K define the subgroup $A(\mathsf{K})$ of A by

$$A(\mathsf{K}) = \{a \in A : \mathsf{H}(a) \geq \mathsf{K}\} = \bigcap_{p \in P} p^{\mathsf{K}_p} A .$$

The *height matrix* $\mathbb{H}_A(a)$ of an element a in A is defined

$$\mathsf{H}_A(a) = \begin{bmatrix} h_2^A(a),\ h_2^A(2a),\ \ldots,\ h_2^A(2^k a),\ \ldots \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ h_p^A(a),\ h_p^A(pa),\ \ldots,\ h_p^A(p^k a),\ \ldots \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{bmatrix} = [\sigma_{pk}] .$$

The element σ_{pk} in the (p, k)-position of $\mathsf{H}_A(a)$ records the p-height of $p^k a$ for all p in P and $k = 0, 1, \ldots$. We shall often write just $\mathsf{H}(a)$ when there is no ambiguity. For a discussion of the height matrix and its fundamental properties, see Fuchs [2], especially p. 198.

A *height matrix* $\mathsf{M} = [\sigma_{pk}]$, $p \in P$, $k = 0, 1, \ldots$ is an $\omega \times \omega$ matrix whose entries σ_{pk} are ordinals and symbols ∞ satisfying

$$\sigma_{p,k+1} \geq \sigma_{pk} + 1$$

for all p and k . Fuchs [2] has shown that every height matrix occurs as the height matrix of an element in some group.

Given a height matrix $\mathsf{M} = [\sigma_{pk}]$ and a prime p define $p\mathsf{M}$ to be the matrix with p-row $(\sigma_{p1}, \sigma_{p2}, \ldots)$ and all other rows the same as for M . Thus 'multiplication' of M by p shifts the p-row 'one-to-the-left'. For arbitrary positive integers n and k we define $(nk)\mathsf{M} = n(k\mathsf{M})$; this definition together with the above definition of $p\mathsf{M}$ for each prime p yields a 'multiplication' of height matrices by arbitrary positive integers. Two height matrices M and N are said to be *equivalent* (we write $\mathsf{M} \sim \mathsf{N}$) if there are positive integers m and n such that $m\mathsf{M} = n\mathsf{N}$. Clearly $\sim$ is an equivalence relation and we denote the *equivalence class* of the height matrix M by $\tilde{\mathsf{M}}$. Height matrices were first introduced by Rotman [1] who called them 'Ulm towers' and their equivalence classes 'Ulm castles'.

Sometimes it is more convenient to consider just one row of a height matrix; if p is a prime then the *p-indicator* of an element a in A is the p-row of $\mathsf{H}(a)$ and is denoted $U_p^A(a)$. For a height matrix $\mathsf{M} = [\sigma_{pk}]$ and prime p we define

$$\mathsf{M}_p = (\sigma_{p0}, \sigma_{p1}, \ldots) .$$

When $\mathsf{H}(a) = \mathsf{M}$ it follows that $U_p(a) = \mathsf{M}_p$ for each prime p. A *p-indicator* is a sequence

$$\mathsf{u}_p = (\sigma_0, \sigma_1, \ldots)$$

of ordinals and symbols ∞ such that $\sigma_i + 1 \leq \sigma_{i+1}$ for $i = 0, 1, \ldots$. Multiplication of p-indicators by powers of p and equivalence of p-indicators are defined by modifying the previous definitions for height matrices in the obvious way. Suppose we are given a group A, an element a in A and a p-indicator $\mathsf{u} = (\sigma_0, \sigma_1, \ldots)$. If $U_p(a) = (\sigma_n, \sigma_{n+1}, \ldots)$ then it is easy to see that there is an element a' in A such that $p^n a' = a$ and $U_p(a) = (\sigma_0', \sigma_1', \ldots, \sigma_{n-1}', \sigma_n, \sigma_{n+1}, \ldots)$ such that $\sigma_i' \geq \sigma_i$ for $0 \leq i < n$.

When dealing with p-indicators for some fixed prime p, the reference to p may be dropped. An indicator $\mathsf{u} = (\sigma_0, \sigma_1, \ldots)$ has a *gap* if, for some $k \geq 0$, we have $\sigma_k + 1 < \sigma_{k+1}$; in this case the gap is said to *follow* σ_k and to *precede* σ_{k+1}. For notational convenience we say that a gap precedes σ_0 - however, we do not consider this gap to be in u. A height matrix has a gap if one of its rows has a gap. From this point on, all unexplained notation or terminology applied to indicators will be taken as obvious from the appropriate definition for height matrices.

Let $\mathsf{M} = [\sigma_{pk}]$ and $\mathsf{N} = [\rho_{pk}]$ be height matrices. Then we write $\mathsf{M} \geq \mathsf{N}$ to mean $\sigma_{pk} \geq \rho_{pk}$ for all p in P and $k = 0, 1, \ldots$. For each group A and height matrix M define

$$A(\mathsf{M}) = \{a \in A : \mathsf{H}(a) \geq \mathsf{M}\} = \bigcap_{p \in P} A(\mathsf{M}_p) .$$

Multiplication of height matrices by positive integers is compatible with 'multiplication' of groups by such integers in the sense of the following lemma.

2.3 LEMMA. *Let* A *be a group and let* $\mathfrak{M}$ *be a height matrix. Then*

$$m[A(\mathfrak{M})] = A(m\mathfrak{M})$$

for every positive integer m .

Proof. Reduction to the case where $m = p$ is immediate. The inclusion $p[A(\mathfrak{M})] \leq A(p\mathfrak{M})$ is trivial; for the opposite inclusion, let the row of $\mathfrak{M}$ corresponding to p be $(\sigma_0, \sigma_1, \ldots)$ and suppose $a \in A(p\mathfrak{M})$. Then $h_p^A(a) \geq \sigma_1$. There is an x in A such that $px = a$ and $h_p(x) \geq \sigma_0$. Thus $x \in A(\mathfrak{M})$ and $a = px \in p[A(\mathfrak{M})]$. $\square$

Given a height matrix $\mathfrak{M}$ we define a function $S_{\mathfrak{M}}$ which associates the subgroup $A(\mathfrak{M})$ with each A . Observe that $A(\mathfrak{M})$ is a fully invariant subgroup of A and further, $A(\mathfrak{M})$ is functorial in the sense that every homomorphism $\alpha : A \to B$ maps $A(\mathfrak{M})$ into $B(\mathfrak{M})$. Thus the function $S_{\mathfrak{M}} : A \mapsto A(\mathfrak{M})$ is a preradical. We will later show that for each height matrix $\mathfrak{M}$ the function $S_{\mathfrak{M}}$ is a cotorsion functor which has enough projectives exactly when $\mathfrak{M}$ has no gaps. Note that when $\mathfrak{M}$ has no gaps, $A(\mathfrak{M}) = A(\mathsf{K})$ for every group A where K is the height formed by the first column of $\mathfrak{M}$. Conversely, to each height $\mathsf{K} = \langle\langle \beta_2, \beta_3, \ldots, \beta_p, \ldots\rangle\rangle$ there corresponds the height matrix $\mathfrak{M}$ whose p-row is $(\beta_p, \beta_p+1, \beta_p+2, \ldots)$ for all p in P and which satisfies $A(\mathsf{K}) = A(\mathfrak{M})$ for every group A . Thus the functor $S_{\mathsf{K}} : A \mapsto A(\mathsf{K})$ associated with the height K has enough projectives. Similarly, if $\mathsf{u} = (\sigma_0, \sigma_1, \ldots)$ is a p-indicator then the height matrix $\mathfrak{M}$ whose p-row is $(\sigma_0, \sigma_1, \ldots)$ and whose remaining rows are all $(0, 1, 2, 3, \ldots)$ satisfies $A(\mathfrak{M}) = A(\mathsf{u})$ for every A . The functors $S_{\mathfrak{M}}$, S_{K} and S_{u} all generalise the well known functors $S_{p^\sigma} : A \to p^\sigma A$.

The remainder of this section details a number of properties of the subgroups $A(\mathsf{M})$.

Given a height matrix M and a set $\{A_i : i \in I\}$ of groups , it is clear that

$$\left(\prod_{i\in I} A_i\right)(\mathsf{M}) = \prod_{i\in I} A_i(\mathsf{M}) \quad\text{and}\quad \left(\bigoplus_{i\in I} A_i\right)(\mathsf{M}) = \bigoplus_{i\in I} A_i(\mathsf{M}) .$$

The next lemma will be needed on several occasions. For a proof see Lemma 37.1 of Fuchs [1].

2.4 LEMMA. *Let* $\alpha : A \to B$ *be an epimorphism such that* $\ker \alpha$ *contains only elements of* p*-height* $\geq \sigma$. *Then*

$$h_p(\alpha a) = h_p(a)$$

whenever $h_p(a) < \sigma$. □

2.5 DEFINITION. Let $u = (\sigma_0, \sigma_1, \ldots)$ be an indicator and define

$$\tau(u) = \sup\{\sigma_i : \text{a gap precedes } \sigma_i\} .$$

If M is a height matrix then define $\tau(\mathsf{M})$ to be the height

$$\tau(\mathsf{M}) = \langle\langle \tau(\mathsf{M}_2), \ldots, \tau(\mathsf{M}_n), \ldots\rangle\rangle . \qquad □$$

2.6 THEOREM. *Let* M *be a height matrix. If* K *is a height such that* $\mathsf{K} \geq \tau(\mathsf{M})$ *and* B *is a subgroup of* $A(\mathsf{K})$ *then*

(i) $B \leq A(\mathsf{M})$;

(ii) $A(\mathsf{M})/B = (A/B)(\mathsf{M})$; *and*

(iii) $\big(A/A(\mathsf{M})\big)(\mathsf{K}) = 0$.

In particular, $A/A(\mathsf{M})$ *is always reduced.*

Proof. The first statement is trivial while the second follows directly from (2.4). Since $A(\mathsf{M}) = \bigcap_{p\in P} A(\mathsf{M}_p)$ and therefore $A/A(\mathsf{M}) \leq \prod_{p\in P} A/A(\mathsf{M}_p)$

then $(A/A(\mathsf{M}))(K) = 0$ will follow once we have shown $p^{\tau(\mathsf{M}_p)}(A/A(\mathsf{M}_p)) = 0$ for all primes p. Let $\mathsf{M}_p = \mathsf{u} = (\sigma_0, \sigma_1, \ldots)$. Two cases arise.

Case I: $\mathsf{u} = (\sigma_0, \sigma_1, \ldots)$ has only a finite number of gaps. We begin by showing that for every A with $p^\sigma A = 0$ and ordinal ρ less than σ, we have $p^\sigma(A/p^\rho A[p]) = 0$. To see this, note that the exact sequence $0 \to A[p] \to A \to pA \to 0$ yields $p^\sigma(A/A[p]) = 0$, so that exactness of

$$0 \to A[p]/p^\rho A[p] \to A/p^\rho A[p] \to A/A[p] \to 0$$

implies $p^\sigma(A/p^\rho A[p]) \leq A[p]/p^\rho A[p]$. Thus if

$$a + p^\rho A[p] \in p^\sigma(A/p^\rho A[p])$$

then we may assume $pa = 0$ while (2.4) implies $a \in p^\rho A$. But then $a \in p^\rho A[p]$. Hence $p^\sigma(A/p^\rho A[p]) = 0$.

As u has only finitely many gaps, $\tau(\mathsf{u}) = \sigma_n$ for some n. Then $A(p^n\mathsf{u}) = p^{\sigma_n}A$ and (2.4) implies $p^{\sigma_n}(A/p^{\sigma_n}A) = 0$; in particular, when $n = 0$ we have shown $p^{\tau(\mathsf{u})}(A/A(\mathsf{u})) = 0$. For the case $n > 0$ let $G = A/A(p^n\mathsf{u})$ and $H = A(p^{n-1}\mathsf{u})/A(p^n\mathsf{u})$ so that $H = p^{\sigma_{n-1}}G[p]$. Since $p^{\sigma_n}G = 0$, it follows that $p^{\sigma_n}(A/A(p^{n-1}\mathsf{u})) = p^{\sigma_n}(G/H) = 0$. Induction yields $p^{\sigma_n}(A/A(\mathsf{u})) = p^{\tau(\mathsf{u})}(A/A(\mathsf{u})) = 0$.

Case II: u has an infinite number of gaps. Then $\tau(\mathsf{u}) = \sup \sigma_i$. Set

$$\mathsf{u}_n = (\sigma_0, \ldots, \sigma_n, \sigma_n+1, \sigma_n+2, \ldots)$$

so that $A(\mathsf{u}) = \bigcap\limits_{n<\omega} A(\mathsf{u}_n)$. Now $A/A(\mathsf{u}) \leq \prod\limits_{n<\omega} A/A(\mathsf{u}_n)$ and $p^{\sigma_n}(A/A(\mathsf{u}_n)) = 0$ for every n so that $p^{\tau(\mathsf{u})}(A/A(\mathsf{u})) = 0$. □

Given two height matrices M and N we can form the composite $S_{\mathsf{N}} \circ S_{\mathsf{M}}$ of their associated functors. In what follows, we show that $S_{\mathsf{N}} \circ S_{\mathsf{M}}$ is also associated with a height matrix. We do this by first defining a 'sum' height matrix $\mathsf{M} + \mathsf{N}$ and then showing that $[A(\mathsf{M})](\mathsf{N}) = A(\mathsf{M} + \mathsf{N})$ for every group A ; thus $S_{\mathsf{N}} \circ S_{\mathsf{M}} = S_{\mathsf{M}+\mathsf{N}}$.

2.7 DEFINITION. Let $\mathsf{u} = (\sigma_0, \sigma_1, \ldots)$ be an indicator, and let ρ be an ordinal or ∞ . If ρ is infinite let γ be that (unique) ordinal or symbol ∞ for which $\omega + \gamma = \rho$. We define

$$\mathsf{u} * \rho = \begin{cases} \sigma_\rho & \text{if } \rho \text{ is finite;} \\ \\ \left(\lim\limits_{i<\omega} \sigma_i\right) + \gamma & \text{if } \rho \text{ is infinite.} \end{cases}$$

If $\mathsf{v} = (\rho_0, \rho_1, \ldots)$ is also an indicator we set

$$\mathsf{u} + \mathsf{v} = (\mathsf{u}*\rho_0, \mathsf{u}*\rho_1, \ldots)$$

and if M and N are height matrices then we define $\mathsf{M} + \mathsf{N}$ to be the height matrix whose p-row is $\mathsf{M}_p + \mathsf{N}_p$ for all p in P . □

For every group A and height matrix M there is a height matrix M' not containing ∞ such that $A(\mathsf{M}) = A(\mathsf{M}')$ so for the remainder of this section we assume that all height matrices and indicators do not contain ∞ .

2.8 LEMMA. *Let* ρ *be an infinite ordinal and let* $\mathsf{u} = (\sigma_0, \sigma_1, \ldots)$ *be an indicator. Then*

$$p^\rho[A(\mathsf{u})] = p^{\mathsf{u}*\rho}A .$$

Proof. Consider the case $\rho = \omega$. We have

$$p^\omega[A(\mathsf{u})] = \bigcap_{n<\omega} p^n[A(\mathsf{u})] = \bigcap_{n<\omega} [A(p^n\mathsf{u})]$$

$$\leq \bigcap_{n<\omega} p^{\sigma_n}A \quad = p^{\mathsf{u}*\omega}A$$

while $p^{u*\omega}A \le p^n[A(u)]$ for all n less than ω implies

$$p^{u*\omega}A \le \bigcap_{n<\omega} p^n[A(u)] = p^\omega[A(u)] .$$

For an arbitrary infinite ordinal ρ we have

$$p^\rho[A(u)] = p^\gamma\big(p^\omega[A(u)]\big) = p^{u*\omega+\gamma}A = p^{u*\rho}A . \qquad \square$$

Thus, if ρ is an arbitrary ordinal or ∞ and u is a p-indicator then combining (2.3) and (2.8) we have:

2.9 PROPOSITION. *For every group* A ,

$$p^\rho[A(u)] = A\big(u*\rho,\ u*(\rho+1),\ u*(\rho+2),\dots\big) . \qquad \square$$

2.10 THEOREM. *If* u *and* v *are indicators then*

$$[A(u)](v) = A(u+v) .$$

Proof. Suppose $u = (\sigma_0, \sigma_1, \dots)$ and $v = (\rho_0, \rho_1, \dots)$. If $a \in [A(u)](v)$ then $p^n a \in p^{\rho_n}[A(u)]$ for $n = 0, 1, \dots$. Now $p^{\rho_n}[A(u)] \le p^{u*\rho_n}A$ for each n , so that $a \in A(u+v)$. On the other hand, when $a \in A(u+v) = A\big(u*\rho_0,\ u*\rho_1, \dots\big)$ we have $p^n a \in A\big(u*\rho_n,\ u*\rho_{n+1}, \dots\big)$ for $n = 0, 1, \dots$. Since $\rho_n+1 \le \rho_{n+1}$ for $n = 0, 1, \dots$ then $u * \big(\rho_n+1\big) \le u * \rho_{n+1}$ and

$$\begin{aligned} A\big(u*\rho_n,\ u*\rho_{n+1}, \dots\big) &\le A\big(u*\rho_n,\ u*(\rho_n+1), \dots\big) \\ &= p^{\rho_n}[A(u)] . \end{aligned}$$

Thus $a \in [A(u)](v)$. $\square$

2.11 THEOREM. *Let* M *and* N *be height matrices. Then*

$$[A(\mathsf{M})](\mathsf{N}) = A(\mathsf{M}+\mathsf{N}) .$$

Before we can prove this theorem we need two vital manipulative lemmas.

2.12 LEMMA. *Let* u_p *and* v_q *be* p- *and* q-*indicators respectively. If* $p \neq q$ *then*

$$[A(u_p)](v_q) = A(u_p) \cap A(v_q) .$$

Proof. The inclusion $[A(u_p)](v_q) \leq A(u_p) \cap A(v_q)$ is obvious. As a first step toward the opposite inclusion we induct on σ to show that $A(u_p) \cap q^\sigma A = q^\sigma[A(u_p)]$. The equality holds when $\sigma = 0$. Suppose we have shown $A(u_p) \cap q^\rho A = q^\rho[A(u_p)]$ whenever $\rho < \sigma$. If σ is a limit ordinal then

$$q^\sigma[A(u_p)] = \bigcap_{\rho<\sigma} q^\rho[A(u_p)] = \bigcap_{\rho<\sigma} \left(A(u_p) \cap q^\rho A\right) = A(u_p) \cap q^\sigma A .$$

When $\rho + 1 = \sigma$ and $a \in A(u_p) \cap q^{\rho+1}A$ then $a = qx$, where $x \in q^\sigma A$. However, $a \in A(u_p)$ implies $x \in A(u_p)$. Then for each n such that $n \geq 0$ it follows that $q^n a \in A(u_p) \cap q^{\sigma n}A = q^{\sigma n}[A(u_p)]$ so $a \in [A(u_p)](v_q)$. □

2.13 LEMMA. *If* M *is a height matrix and* u_p *is a* p-*indicator then*

$$[A(\mathsf{M})](u_p) = \bigcap_{\substack{q\in P \\ q\neq p}} A(\mathsf{M}_q) \cap [A(\mathsf{M}_p)](u_p) .$$

Proof. It clearly suffices to prove that

$$p^\rho[A(\mathsf{M})] = \bigcap_{\substack{q\in P \\ q\neq p}} [A(\mathsf{M}_q)] \cap p^\rho[A(\mathsf{M}_p)] \tag{3}$$

for each prime p and ordinal ρ . If $\rho = 0$ then (3) is satisfied trivially. Suppose that σ is an arbitrary ordinal and that (3) holds when $\rho < \sigma$. If σ is a limit ordinal then

$$p^{\sigma}[A(\mathsf{M})] = \bigcap_{\rho<\sigma} p^{\rho}[A(\mathsf{M})] = \bigcap_{\rho<\sigma}\left(\bigcap_{\substack{q\in P\\ q\neq p}} A(\mathsf{M}_q)\cap p^{\rho}[A(\mathsf{M}_p)]\right)$$

$$= \left(\bigcap_{\substack{q\in P\\ q\neq p}} A(\mathsf{M}_q)\right) \cap \left(\bigcap_{\rho<\sigma} p^{\rho}[A(\mathsf{M}_p)]\right) = \bigcap_{\substack{q\in P\\ q\neq p}} A(\mathsf{M}_q) \cap p^{\sigma}[A(\mathsf{M}_p)] .$$

For the remaining case where $\sigma = \rho + 1$ we remark that (3) is equivalent to

$$p^{\rho}[A(\mathsf{M})] = A(\mathsf{M}^{\rho})$$

where

$$\mathsf{M}_p^{\rho} = (\mathsf{M}_p{*}\rho,\ \mathsf{M}_p{*}(\rho+1),\ \ldots) \quad \text{and} \quad \mathsf{M}_q^{\rho} = \mathsf{M}_q \quad \text{for} \quad q \neq p .$$

Thus

$$p^{\sigma}[A(\mathsf{M})] = p(p^{\rho}[A(\mathsf{M})])$$
$$= p[A(\mathsf{M}^{\rho})]$$
$$= A(p\mathsf{M}^{\rho}) \quad \text{by (2.3)}$$
$$= A(\mathsf{M}^{\sigma}) . \qquad \square$$

Proof of 2.11. Recalling that $A(\mathsf{M}) = \bigcap_{p\in P} A(\mathsf{M}_p)$ we have

$$[A(\mathsf{M})](\mathsf{N}) = \bigcap_{p\in P} [A(\mathsf{M})](\mathsf{N}_p)$$
$$= \bigcap_{p\in P}\bigcap_{q\in P} [A(\mathsf{M}_q)](\mathsf{N}_p) \quad \text{by (2.13)}$$
$$= \bigcap_{p\in P} [A(\mathsf{M}_p)](\mathsf{N}_p) \quad \text{by (2.12)}$$
$$= \bigcap_{p\in P} A(\mathsf{M}_p+\mathsf{N}_p) \quad \text{by (2.10)}$$
$$= A(\mathsf{M} + \mathsf{N}) . \qquad \square$$

We close this section with the remark that a subgroup B of A is isotype in A exactly when $A(\mathsf{M}) \cap B = B(\mathsf{M})$ for every height matrix M ; it follows from (2.11) that $B(\mathsf{M})$ is isotype in $A(\mathsf{M})$ for every height matrix M .

Totally projective groups and associated results

A group A is said to be p^σ*-projective* if $p^\sigma \operatorname{Ext}(A, X) = 0$ for all groups X, and A is said to be *totally projective* if $A/p^\sigma A$ is p^σ-projective for every ordinal σ and prime p (Nunke [3]).

An exact sequence $0 \to B \to A \to C \to 0$ of p-groups is said to be *balanced* (Fuchs [2]) if the sequence $0 \to p^\sigma B \to p^\sigma A \to p^\sigma C \to 0$ is exact for every ordinal σ; in this case we say that B, considered as a subgroup of A, is balanced in A. Observe that B is balanced in A exactly when B is both isotype and p-nice in A. In Chapter 3, we define balanced sequences of arbitrary groups and show that in the case of p-groups, our definition coincides with the above definition given by Fuchs.

The p-group A has a *nice composition series* (Fuchs [2]) if there is a well ordered ascending chain

$$0 = N_0 < N_1 < \ldots < N_\lambda < \ldots < N_\mu = A$$

of subgroups of A such that:

(a) $N_0 = 0$ and $N_\mu = A$;

(b) each N_λ is p-nice in A;

(c) $|N_{\lambda+1} : N_\lambda| = p$ for every $\lambda < \mu$; and

(d) $N_\lambda = \bigcup_{\sigma<\lambda} N_\sigma$ if λ is a limit ordinal.

For each ordinal σ and prime p Nunke [1] constructs a group H^p_σ satisfying

(i) $l_p(H^p_\sigma) = \sigma$;

(ii) $p^\sigma H^p_{\sigma+1}$ is cyclic of order p, and $H_{\sigma+1}/p^\sigma H_{\sigma+1} \cong H_\sigma$; and

(iii) $H^p_\sigma = \bigoplus_{\rho<\sigma} H^p_\rho$ if σ is a limit ordinal.

The H^p_σ are determined up to isomorphism by conditions (i)-(iii) and are known as the *generalised Prüfer groups*. We also define H^p_∞ isomorphic to $Z(p^\infty)$ for each prime p. The generalised Prüfer groups were first introduced by Nunke in order to construct extensions

$$\mathrm{e}(p, \sigma) : 0 \to Z \to G^p_\sigma \to H^p_\sigma \to 0$$

such that $\mathrm{e}(p, \sigma)$ represents the functor p^σ.

Nunke [3] has shown that a totally projective group is the direct sum of a free group and a totally projective torsion group. Since the free groups introduce only a trivial perturbation, we ignore them so that by a *totally projective group* we will always mean a torsion group. Note that the class of totally projective groups includes the class of divisible torsion groups.

The following are equivalent for a reduced p-group A :

1. *A has a nice composition series;*
2. *A is a direct summand of a direct sum of generalised Prüfer groups;*
3. *A is totally projective; and*
4. *A has the projective property with respect to balanced exact sequences of p-groups.*

The equivalence of 1-4 is due collectively to Nunke, Hill and Fuchs. Hill has also shown that there are 'enough' balanced projectives in the sense that every p-group A can be imbedded in a balanced exact sequence $0 \to G \to H \to A \to 0$ of p-groups with H totally projective. In fact H can be chosen as a direct sum of generalised Prüfer groups. The proofs of these results all appear in Fuchs [2].

Let A and B be groups. We say that a homomorphism α of a subgroup C of A into B *does not decrease heights* (in A) if $H_A(c) \le H_B(\alpha c)$ for every c in C.

2.14 THEOREM (Hill [1], E.A. Walker [1]). *Let A and C be groups* *and let* ϕ *be an isomorphism between a* p*-nice subgroup* G *of* A *and a subgroup* H *of* C *which does not decrease heights. Suppose that*

(i) A/G *is a totally projective* p*-group, and*

(ii) *if* σ *is an ordinal or* ∞ *then* $f_\sigma^p(A, G) \leq f_\sigma^p(C, H)$.

Then ϕ *extends to a monomorphism* ϕ^* *of* A *into* C . *If equality holds in (ii) for every* σ *then* ϕ^* *can be chosen as an isomorphism of* A *onto* C . □

In particular, two totally projective groups are isomorphic if and only if they have the same Ulm invariants. The proof of Corollary 81.4 in Fuchs [2] applies directly to give:

2.15 COROLLARY. *Let* A *and* C *be groups and* η *a homomorphism of a* p*-nice subgroup* G *of* A *into* C *which does not decrease heights. If* A/G *is a totally projective* p*-group then* η *can be extended to a homomorphism* $\eta^* : A \to C$. □

A function g from the ordinals to the cardinals is said to be *admissible* if the following conditions are satisfied:

(i) $\sup\{\sigma+1 : g(\sigma) \neq 0\}$ exists; and

(ii) $\sum_{\rho \geq \sigma+\omega} g(\rho) \leq \sum_{n<\omega} g(\sigma+n)$ for all σ .

Denote the supremum in (i) by $l(g)$ - we call $l(g)$ the *length* of g .

2.16 THEOREM (Crawley and Hales [1], Hill [1]). *Let* g *be a function from the ordinals to the cardinals. There exists a totally projective* p*-group* A *such that* $f_\sigma^p(A) = g(\sigma)$ *for all ordinals* σ *if and only if* g *is admissible.* □

Thus, given a prime p , there is a one-to-one correspondence between admissible functions and isomorphism classes of totally projective p-groups.

2.17 PROPOSITION. *Given two admissible functions* g *and* h, *the totally projective* p*-group* A *corresponding to* g *has a direct summand isomorphic to the totally projective* p*-group* B *corresponding to* h *if and only if the following two conditions are satisfied:*

(i) $g(\sigma) \geq h(\sigma)$ *for every ordinal* σ; *and*

(ii) for every ordinal σ, EITHER $\sigma + \omega > l(g)$ OR *there is a* ρ *such that* $\sigma \leq \rho < \sigma+\omega$ *and* $g(\rho)$ *is infinite* OR *the set* $\{\rho : \sigma \leq \rho < \sigma+\omega$ *and* $g(\rho) > h(\rho)\}$ *is infinite.*

Proof. Suppose g and h satisfy *(i)* and *(ii)*. We define a function k from the ordinals to the cardinals by

$$k(\sigma) = \begin{cases} g(\sigma) & \text{if } g(\sigma) \text{ is infinite; and} \\ g(\sigma) - h(\sigma) & \text{otherwise.} \end{cases}$$

Clearly k is admissible and $g(\sigma) = h(\sigma) + k(\sigma)$ for every σ. Thus the totally projective group C corresponding to k satisfies $A \cong B \oplus C$. The converse is obvious. □

2.18 PROPOSITION. *Let* A *be a reduced totally projective* p*-group and* σ *an ordinal such that* $\sigma \leq l_p(A)$ *and such that if* $\sigma - 1$ *exists then* $f^p_{\sigma-1}(A) \neq 0$. *Then* A *has a summand of length* σ.

Proof. For each ordinal ρ, let $\bar{\rho}$ be the largest non-successor ordinal less than or equal to ρ. Let $g(\rho) = f^p_\rho(A)$ and

$$N(\rho) = \{\gamma : \bar{\rho} \leq \gamma < \rho \text{ and } g(\gamma) \text{ is finite and non-zero}\}.$$

Then the function $h(\rho)$ given by

$$h(\rho) = \begin{cases} g(\rho) & \text{if either } \rho + 1 = \sigma \\ & \quad \text{or } \rho + 1 < \sigma \text{ and either } g(\rho) \text{ is infinite} \\ & \qquad \text{or } |N(\rho)| \text{ is even;} \\ 0 & \text{otherwise} \end{cases}$$

satisfies the conditions of (2.17) and clearly $l(h) = \sigma$. □

Parker and Walker [1] proved that a p-group A is totally projective if and only if A belongs to the smallest class of groups that contains $Z(p)$, is closed under taking direct sums and direct summands, and which contains a group G exactly if it contains both $p^{\sigma}G$ and $G/p^{\sigma}G$, for any ordinal σ. This has been partially extended in:

2.19 PROPOSITION (L. Fuchs and E.A. Walker). *If A is a totally projective p-group and u is an indicator then both $A(\mathsf{u})$ and $A/A(\mathsf{u})$ are totally projective.* □

The proof appears as exercise 7 on p. 100 of Fuchs [2].

Warfield [1] introduced the following class of torsion groups which is slightly larger than the class of totally projective groups. Let λ be a limit ordinal which is not cofinal with ω. Then a p-group A is said to be a *λ-elementary S-group* if there is a totally projective p-group H of length λ such that A is isotype and λ-dense in H and $H/A \cong Z(p^{\infty})$. We have seen in (2.2) that every totally projective p-group of length λ contains such a subgroup. The group A is said to be an *S-group* if A is the direct sum of a totally projective group and λ-elementary S-groups for various primes p and limit ordinals λ not cofinal with ω. We denote the class of S-groups by $\mathcal{S}$.

The class of S-groups enjoys the same closure properties that we have mentioned for totally projective groups: Warfield [1] has shown that for every ordinal σ, the p-group A is an S-group if and only if both $p^{\sigma}A$ and $A/p^{\sigma}A$ are S-groups. Toward the analogue of (2.19) we have:

2.20 THEOREM. *If A is an S-group and u is an indicator then $A(\mathsf{u})$ is an S-group.*

Proof. We need only consider the case where A is a λ-elementary S-group for some limit ordinal λ not cofinal with ω. Let

$u = (\sigma_0, \sigma_1, \ldots)$. If $\sigma_i \geq \lambda$ for some i then $A(u)$ is bounded and therefore trivially an S-group. The remaining possibility is $\sigma_i < \lambda$ for all i . Let H be totally projective such that A is λ-dense and isotype in H and $H/A \cong Z(p^\infty)$. If ρ is the (unique) limit ordinal such that $\tau(u) + \rho = \lambda$ then set $\gamma = \omega + \rho$; we have $\gamma = l_p(A(u)) = l_p(H(u))$ and γ is not cofinal with ω . The concluding remark of section 1 shows that $A(u)$ is isotype in $H(u)$ so it remains to prove that $A(u)$ is γ-dense in $H(u)$ and $H(u)/A(u) \cong Z(p^\infty)$. From $0 \neq p^{\tau(u)}H \leq H(u)$ we have $H = p^{\tau(u)}H + A \leq H(u) + A = H$. Hence

$$H(u)/A(u) \cong (H(u)+A)/A = H/A \cong Z(p^\infty) .$$

For each ordinal σ such that $\sigma < \gamma$ it follows that

$$\begin{aligned}(A(u)+p^\sigma[H(u)])/A(u) &\cong p^\sigma[H(u)]/p^\sigma[A(u)] \\ &= H(v)/A(v)\end{aligned}$$

where $v = (u*\sigma, u*(\sigma+1), \ldots)$. Noting that $\tau(v) < \lambda$ it is clear that the argument used for u also shows $H(v)/A(v) \cong Z(p^\infty)$. Thus

$$Z(p^\infty) \cong H(u)/A(u) \geq (A(u)+p^\sigma[H(u)])/A(u) \cong Z(p^\infty)$$

so that $H(u) = A(u) + p^\sigma[H(u)]$. □

In Chapter 6 it will be shown that if A is an S-group and u is an indicator then $A/A(u)$ is an S-group; thus (2.19) extends to S-groups.

The next lemma is a useful tool for extending homomorphisms.

2.21 LEMMA. *Let A and B be groups and let H and G_i , $i \in I$ be subgroups of A such that $A = \sum_{i \in I} G_i$ and $G_i \cap \sum_{\substack{j \in I \\ j \neq i}} G_j = H$ for all i in I . If $\phi_i : G_i \to B$ are homomorphisms satisfying $\phi_i|_H = \phi_j|_H$ for all i, j in I then there exists a homomorphism $\phi^* : A \to B$ such that $\phi^*|_{G_i} = \phi_i$ for all i in I .*

Proof. Denote the common restriction of the ϕ_i to H by ϕ. It is clear that if $a = \sum g_i$ such that $g_i \in G_i$ is an expression for an element a of $A = \sum_{i \in I} G_i$ then $\phi^* a = \sum \phi_i g_i$ is the only possible common extension of the ϕ_i and we need only show that this ϕ^* is well defined. Letting $a = \sum g_i'$ where $g_i' \in G_i$ be another expression for a we have $g_i - g_i' \in H$ for each i and

$$\begin{aligned}\sum \phi_i g_i - \sum \phi_i g_i' &= \sum \phi(g_i - g_i') \\ &= \phi\left[\sum (g_i - g_i')\right] \\ &= 0 . \qquad \square\end{aligned}$$

2.22 DEFINITION. Let B be a subgroup of A and p a prime. Then we denote by $A(p, B)$ the subgroup of A defined by

$$A(p, B) = \{a \in A : p^k a \in B \text{ for some integer } k \geq 0\} . \qquad \square$$

Observe that $A(p, B)$ is the full inverse image of $(A/B)_p$ under the natural epimorphism $A \to A/B$. Clearly $A(p, B)_p = A_p$ and when $T(B) = B_p$ it follows that $A(p, B)_q = 0$ for all primes $p \neq q$.

2.23 THEOREM. *Let A and C be groups and let η be a homomorphism of a subgroup B of A into C which does not decrease heights. If B is p-nice in $A(p, B)$ for every prime p and A/B is totally projective then η can be extended to a homomorphism $\eta^* : A \to C$.*

Proof. Since B is p-nice in $A(p, B)$ for every prime p then by (2.15) there are homomorphisms $\eta_p : A(p, B) \to C$ which extend η. Now apply (2.21) to get η^*. $\square$

The next theorem provides ample opportunity to apply (2.23).

2.24 LEMMA (Rotman [1], Megibben [1], Wallace [1]). *If* A *has torsion free rank* 1 *and the element* a *in* A *has infinite order then* $\langle a\rangle$ *is* p*-nice in* $A(p, \langle a\rangle)$ *for every prime* p . □

The following is such an application - the proof is straightforward.

2.25 COROLLARY (to 2.23). *Let* g *be an element of* G *having infinite order and suppose that*

$$\mathsf{e} : 0 \to \langle g\rangle \to G \to H^* \to 0$$

is exact with H^* *totally projective. If* $\mathrm{H}_G(g) = \mathrm{M}$ *then* e *represents the functor* S_{M} . □

We conclude with two useful lemmas.

2.26 LEMMA (Wallace [1]). *If* A *has torsion free rank* 1 *and* A_p *is totally projective then* $(A/\langle a\rangle)_p$ *is totally projective for every* a *in* A . □

2.27 LEMMA. *Let* p *be a prime,* 1 *the identity of* Z_p . *Then*

$$h_p^{A}(a) = h_p^{Z_p\otimes A}(1 \otimes a)$$

for each a *in* A .

Proof. For every x in $Z_p \otimes A$ there is an integer n such that $nx \in 1 \otimes A$ and $(n, p) = 1$. Thus if $px = 1 \otimes a$ can be solved in $Z_p \otimes A$ then $pnx = n(1 \otimes a)$ solves $py = n(1 \otimes a)$ in $1 \otimes A$. However $(n, p) = 1$ then implies that $px = 1 \otimes a$ can also be solved in $1 \otimes A$ and the lemma follows. □

CHAPTER 3

BALANCED SUBGROUPS

The balanced subgroups defined by Fuchs [2] in the context of p-groups play an important role in the theory of totally projective p-groups. In this chapter we define balanced subgroups of arbitrary Abelian groups, together with a weaker concept which we call H-balanced. In particular, balanced subgroups are always H-balanced and examples are given of H-balanced subgroups which are not balanced.

The remainder of the chapter is concerned with a general discussion of balanced subgroups and balanced sequences. Although no single result stands out, perhaps the most indispensable is our proof that the new definition of balanced specialises to that of Fuchs.

3.1 DEFINITION. An exact sequence

$$0 \to B \to A \xrightarrow{\alpha} C \to 0 \tag{4}$$

is said to be *balanced* if the induced sequence

$$0 \to B(\mathfrak{M}) \to A(\mathfrak{M}) \to C(\mathfrak{M}) \to 0 \tag{5}$$

is exact for all height matrices $\mathfrak{M}$. When convenient, we instead apply the adjective 'balanced' to the subgroup B, the homomorphism α or the element of $\mathrm{Ext}(C, A)$ corresponding to (4). A group G is *balanced projective* if the induced map $\mathrm{Hom}(G, A) \to \mathrm{Hom}(G, C)$ is surjective for every balanced exact sequence (4), and *balanced injective* if $\mathrm{Hom}(A, G) \to \mathrm{Hom}(B, G)$ is surjective. □

3.2 DEFINITION. An exact sequence $0 \to B \to A \to C \to 0$ is called H-*balanced* if

$$0 \to B(\mathrm{K}) \to A(\mathrm{K}) \to C(\mathrm{K}) \to 0 \tag{6}$$

is exact for every height K. □

The H-balanced projectives and H-balanced injectives are defined in the obvious manner. Observe that $0 \to B \to A \to C \to 0$ is balanced if and only if (5) is exact for just those height matrices $\mathbb{M}$ which occur as the height matrices of elements in A and C. On p. 20 we saw that exactness of (5) at the middle term for all height matrices $\mathbb{M}$ is equivalent to isotypeness of B in A. The same holds for (6) with respect to heights.

A balanced subgroup is clearly H-balanced. Although the converse is not true, we show that if $0 \to B \to A \to C \to 0$ is H-balanced then (5) is exact for a slightly larger class of height matrices than just those corresponding to heights.

3.3 LEMMA. *If* $0 \to B \to A \to C \to 0$ *is* H-*balanced then* (5) *is exact for all height matrices* $\mathbb{M}$ *with finitely many gaps.*

Proof. We first show that if $0 \to B \to A \to C \to 0$ is H-balanced and $0 \to B(p\mathbb{M}) \to A(p\mathbb{M}) \to C(p\mathbb{M}) \to 0$ is exact for a given height matrix $\mathbb{M}$ then (5) is exact for this $\mathbb{M}$. Since B is isotype in A we need only check exactness of (5) at $C(\mathbb{M})$. Suppose $c \in C(\mathbb{M})$. Then $pc \in C(p\mathbb{M})$ and there is an element a_1 in $A(p\mathbb{M})$ such that $\alpha a_1 = pc$. Writing

$\mathbb{M}_p = (\sigma_0, \sigma_1, \ldots)$, we choose a_2 in $p^{\sigma_0}A$ such that $pa_2 = a_1$. Thus $a_2 \in A(\mathbb{M})$ and if $\alpha a_2 = c$ we are done. If not, we choose a_3 in $p^{\sigma_0}A$ such that $\alpha a_3 = c$ (such a choice is possible since we have assumed $0 \to B \to A \to C \to 0$ to be H-balanced). Now

$$pa_3 - pa_2 \in p^{\sigma_0+1}A \cap B = p^{\sigma_0+1}B$$

so there is a b in $p^{\sigma_0}B$ such that $pa_3 - pa_2 = pb$. Clearly $a_3 - b \in A(\mathbb{M})$ so (5) is indeed exact for this $\mathbb{M}$.

The lemma follows on noting that when $\mathbb{M}$ has finitely many gaps, there is an integer m and height K such that $A(m\mathbb{M}) = A(\mathsf{K})$. □

That (3.3) is as far as one can go is illustrated by the following two examples of H-balanced exact sequences which are not balanced - one for each of the two distinct ways in which a height matrix can have infinitely many gaps, namely infinitely many rows having at least one gap or infinitely many gaps in at least one row.

3.4 EXAMPLE. Let

$$A = \prod_{n=1}^{\infty} \langle a_n \rangle \quad \text{where} \quad o(a_n) = p^{2n}$$

and let a be the element

$$\left(pa_1,\ p^2a_2,\ \ldots,\ p^na_n,\ \ldots\right) .$$

Then $U_p(a) = (1, 3, 5, 7, \ldots)$. Now A can be considered as a Z_p-module. Using the same methods as for groups (Fuchs [1], p. 126) one constructs a pure exact sequence

$$0 \to X \to Y \xrightarrow{\eta} A \to 0 \tag{7}$$

where X and Y are direct sums of cyclic Z_p-modules. As groups, X and Y are direct sums of copies of Z_p and cyclic p-groups. The purity of (7) is equivalent to exactness of

$$0 \to p^nX \to p^nY \to p^nA \to 0$$

for all integers n such that $n \geq 0$. Since X and Y have no elements of infinite p-height, and for $q \neq p$ the q-height of each of their elements is ∞ , it follows that (7) is H-balanced. Every y in Y has a p-power multiple either zero or contained in a torsion free summand of Y , so the height matrix of every such y has only finitely many gaps. As $\mathsf{H}(a)$ has infinitely many gaps we conclude that there is no y in Y such that $\eta y = a$ and $U_p(y) = U_p(a)$, so that (7) is not balanced. (Note however that the sequence

$$0 \to X(\mathsf{M}) \to Y(\mathsf{M}) \to A(\mathsf{M}) \to 0 \tag{8}$$

is exact for every height matrix with finitely many gaps in each row.) □

3.5 EXAMPLE. Let T be the torsion part of

$$G = \prod_{p\in P} \left(\langle b_p \rangle \oplus Z_p\right) \quad \text{where} \quad o\left(b_p\right) = p \ .$$

Let x_p be the unit of Z_p and write $a_p = b_p + px_p$. Then the element a such that

$$a = \left(a_p\right)_{p\in P}$$

has infinite order and $U_p(a) = (0, 2, 3, 4, \ldots)$ for every prime p . Choose A in G such that $A/T = \langle a+T \rangle_* \leq G/T$ and let

$$0 \to X \to Y \xrightarrow{\delta} A \to 0 \tag{9}$$

be pure exact with Y a direct sum of cyclic groups (that is, (9) is a pure projective resolution of A). We prove that (9) is H-balanced. As A has torsion free rank 1 , a glance at the height matrix of a shows that the only heights which occur for infinite order elements of A are those which also occur in Z . It follows that if K is a height for which $A(\mathrm{K})$ is not torsion then there is an integer k such that $H(\mathrm{K}) = kH$ for all groups H . Therefore exactness of $0 \to nX \to nY \to nA \to 0$ for every positive integer n ensures that $0 \to X(\mathrm{K}) \to Y(\mathrm{K}) \to A(\mathrm{K}) \to 0$ is exact whenever $A(\mathrm{K})$ is not torsion. It remains only to show that $0 \to T(X) \to T(Y) \to T(A) \to 0$ is H-balanced and this is immediate from the fact that, for every prime p , Y_p and A_p have no non-zero elements of infinite p-height and that $T(X)$ is pure in $T(Y)$. The argument used in (3.4) shows that (9) cannot be balanced. However, in this case the induced sequence (8) is exact for every height matrix $\mathbb{M}$ having gaps in at most a finite number of rows. □

Before examining the properties of balanced subgroups we generalise the concept of a nice subgroup first introduced by Hill [1].

3.6 DEFINITION. Let B be a subgroup of A . An element a in $A\backslash B$ is $\mathbb{H}$-*proper* (H-*proper*) with respect to B if

$$\mathbb{H}_A(a) = \mathbb{H}_{A/B}(a+B) \quad \left(\mathrm{H}_A(a) = \mathrm{H}_{A/B}(a+B)\right) \ .$$

The subgroup B is $\mathbb{H}$-*nice* (H-*nice*) in A if every coset $a + B$ contains an element $\mathbb{H}$-proper (H-proper) with respect to B. □

Thus a subgroup B of A is balanced exactly when B is both isotype and $\mathbb{H}$-nice in A. In fact, definition (3.6) was chosen so that results connecting balanced subgroups, $\mathbb{H}$-nice subgroups and isotype subgroups take the same form for arbitrary groups as the corresponding results (in the context of p-groups) connecting the balanced subgroups of Fuchs with p-nice subgroups and isotype subgroups. A case in point is the following. If B is a subgroup of A then a in A is p-proper with respect to B if and only if a has maximal p-height among the elements of the coset $a + B$. The corresponding result (whose proof follows directly from the preceding statement) for height matrices is:

3.7 LEMMA. *Let B be a subgroup of A. Then an element a of A is $\mathbb{H}$-proper with respect to B if and only if $p^k a$ has maximal p-height in the coset $p^k a + B$ for $k = 0, 1, \ldots$ and all primes p.* □

The next lemma collects some properties of $\mathbb{H}$-nice subgroups. Corresponding statements for H-nice subgroups can be proved with identical arguments and hence are ommitted.

3.8 LEMMA. *Let B and C be subgroups of A such that $C \leq B$. Then:*

(i) if B is $\mathbb{H}$-nice in A then B/C is $\mathbb{H}$-nice in A/C; and

(ii) if C is $\mathbb{H}$-nice in A and B/C is $\mathbb{H}$-nice in A/C then B is $\mathbb{H}$-nice in A.

Proof. (i) For an element a in A, write $\bar{a}$ to denote the coset $a + C$. From the natural isomorphism $A/B \cong (A/C)/(B/C)$ we have

$$\mathbb{H}(\bar{a}+B/C) = \mathbb{H}(a+B) \quad \text{for all } a \text{ in } A\backslash B .$$

Then for each a in $A\backslash B$ choose b in B such that $\mathbb{H}(a+b) = \mathbb{H}(a+B)$;

since a homomorphism does not decrease heights it follows that

$\mathsf{H}(\bar{a}+B/C) \geq \mathsf{H}(a+b+C) \geq \mathsf{H}(a+b)$ and therefore $\mathsf{H}(a+b+C) = \mathsf{H}(\bar{a}+B/C)$, so we have found an element $a+b+C$ in A/C which is H-proper with respect to B/C.

(ii) If $\mathsf{H}(a+B) = \mathsf{M}$ then by assumption there is b in B satisfying $\mathsf{H}(a+b+C) = \mathsf{M}$ and c in C satisfying $\mathsf{H}(a+b+c) = \mathsf{M}$. □

We are now ready to show that Fuchs' definition of a balanced subgroup coincides with ours in the special case of p-groups. Fuchs [2] has shown that the following three conditions are equivalent for a subgroup B of a p-group A (we denote the quotient A/B by C and the natural map $A \to C$ by α):

(a') B *is balanced (in the sense of Fuchs) in* A *;*

(b') $\alpha\left(p^{\sigma}A[p]\right) = p^{\sigma}C[p]$ *for every ordinal* σ *; and*

(c') $0 \to B/p^{\sigma}B \to A/p^{\sigma}A \to C/p^{\sigma}C \to 0$ *is exact for each ordinal* σ.

3.9 LEMMA. *The following are equivalent for a subgroup* B *of a* p*-group* A *(let* $C = A/B$ *and let* $\alpha : A \to C$ *be the natural epimorphism):*

(d) B *is balanced in* A *;*

(e) B *is balanced in* A *in the sense of Fuchs; and*

(f) to every element c *in* C *there is an* a *in* A *such that*

$$\alpha a = c\,, \quad U_p(a) = U_p(c) \quad \text{and} \quad o(a) = o(c)\,.$$

Proof. That *(d)* implies *(e)* is obvious and *(f)* implies *(e)* is immediate from *(b')*.

Assuming *(e)*, we now prove *(f)*. Suppose $c \in C$ and $U_p(c) = (\sigma_0, \sigma_1, \ldots)$. Let $o(c) = p^n$; it follows that $\sigma_n = \infty$. For each integer r such that $0 \leq r < n$ we choose a totally projective group H_r and in that, an x_r of order p^{r+1} as follows. If $\sigma_r = \infty$ then let

$H_r = Z(p^\infty)$ and let x_r be an element of order p^{r+1} in H_r. If $\sigma_r \neq \infty$ let H_r be the generalised Prüfer p-group of length $\sigma_r + 1$ and choose x_r of order p^{r+1} in H_r such that $h_p\left(p^r x_r\right) = \sigma_r$ and such that $h_p\left(p^s x_r\right) \geq \sigma_s$ for integers s for which $0 \leq s < r$ (the possibility of doing this was discussed on p. 13). In $H = \bigoplus_{0 \leq r < n} H_r$ we select $x = x_0 + \ldots + x_{n-1}$ and note that $U_p(c) = U_p(x)$. Now $\langle x \rangle$, being finite, is a p-nice subgroup of H and so the height preserving isomorphism $\langle x \rangle \to \langle c \rangle$ sending $x \longmapsto c$ extends to a homomorphism $\phi : H \to C$ by (2.14). Since H is projective with respect to balanced (in the sense of Fuchs) sequences of p-groups, ϕ lifts to a homomorphism $\psi : H \to A$ such that $\alpha\psi = \phi$. Writing $a = \psi x$ we have $\alpha a = c$, $U_p(a) = U_p(c)$ and $o(a) = o(c)$. Thus *(e)* and *(f)* are equivalent, and we complete the proof by assuming *(e)* and *(f)* together to show *(d)*. Now *(e)* implies B is isotype in A while it is evident from *(f)* that B is H-nice in A. □

Fuchs [2] also defines the concept of a balanced subgroup of a torsion free group - his definition in this case is the same as our definition of an H-balanced subgroup. However, the height matrix of each element in a torsion free group is always free of gaps, so again Fuchs' definition coincides with our definition of a balanced subgroup in this special case. We can say more:

3.10 PROPOSITION. *If* $0 \to B \to A \xrightarrow{\alpha} C \to 0$ *is balanced then the sequences* $0 \to T(B) \to T(A) \to T(C) \to 0$ *and* $0 \to B/T(B) \to A/T(A) \to C/T(C) \to 0$ *are balanced.*

Proof. It follows from *(b')* above that $0 \to T(B) \to T(A) \to T(C) \to 0$ is balanced if and only if

$$\left(p^\sigma T(A)[p]+T(B)\right)/T(B) \geq p^\sigma T(C)[p]$$

for all primes p and ordinals σ. Let p^σ be fixed, and $c \in p^\sigma T(C)[p]$. The $\mathbb{H}$-niceness of B in A allows a choice of a in $p^\sigma A$ such that $\alpha a = c$. Then $pa \in p^{\sigma+1}A \cap B = p^{\sigma+1}B$ as B is isotype in A, so $pa = pb$ for some b in $p^\sigma B$. Now $a - b \in p^\sigma A[p] = p^\sigma T(A)[p]$ and $\alpha(a-b) = c$.

Exactness of $0 \to B/T(B) \to A/T(A) \to C/T(C) \to 0$ follows directly from exactness of $0 \to T(B) \to T(A) \to T(C) \to 0$. Therefore, by (2.1), $B/T(B)$ is isotype in $A/T(A)$ and since the height matrix of an element in a torsion free group is determined by its height, it remains only to show that $B/T(B)$ is H-nice in $A/T(A)$. In preparation for the proof of this, we establish that the p-indicator of an element g in an arbitrary group G determines the p-height of the coset $g + T(G)$ in $G/T(G)$. Namely, if $U_p(g) = (\sigma_0, \sigma_1, \ldots)$ and $\beta : G \to G/T(G)$ is the natural epimorphism then

$$h_p(\beta g) = \begin{cases} \sigma_n - n & \text{if } \sigma_n < \omega \text{ and } \sigma_n + k = \sigma_{n+k} \text{ for } k = 0, 1, \ldots ; \text{ and} \\ \infty & \text{otherwise.} \end{cases} \tag{10}$$

To see this, suppose $\sigma_n < \omega$ and $\sigma_n + k = \sigma_{n+k}$ for $k = 0, 1, \ldots$. Then

$$h_p(p^{n+r}g) = \sigma_n + r . \tag{11}$$

In view of the fact that $h_p(p^n \beta g) \geq \sigma_n$ and that βG is torsion free, we must have $h_p(\beta g) \geq \sigma_n - n$. If $h_p(\beta g) > \sigma_n - n$ then there is a g' in G such that $h_p(\beta g') \geq \sigma_n - n$ and $p\beta g' = \beta g$. Thus $g - pg' \in T(G)$; let $o(g-pg') = p^l r$ where $(p, r) = 1$. Now $p^{n+l}g$ and $p^{n+l+1}g'$ differ by an element of order r, so $h_p(p^{n+l}g) = h_p(p^{n+l+1}g') \geq \sigma_n + l + 1$, contradicting (11). When $U_p(g)$ has infinitely many gaps or has an infinite term (these two possibilities make up the 'otherwise' case), it is clear that $h_p(\beta g) = \infty$.

Now if $c + T(C) \in C/T(C)$ there is an a in A such that $\mathsf{H}(a) = \mathsf{H}(c)$ and $\alpha a = c$. By (10), we must have

$$\mathrm{H}_{A/T(A)}\big(a+T(A)\big) = \mathrm{H}_{C/T(C)}\big(c+T(C)\big) .$$

Thus $B/T(B)$ is H-nice in $A/T(A)$. $\square$

3.11 PROPOSITION. *Let B be a subgroup of A . Then B is balanced in A if (and only if) $T(B)$ is balanced in $T(A)$ and B is H-nice in A .*

Proof. Write $C = A/B$ and let $\alpha : A \to C$ be the natural epimorphism. The bracketed statement is immediate from (3.10). For the rest, assume $T(B)$ is balanced in $T(A)$ and B is H-nice in A : it suffices to prove that B is isotype in A . Now $B \cap p^{\sigma}A \leq p^{\sigma}B$ is trivial when $\sigma = 0$. Suppose we have shown $B \cap p^{\rho}A \leq p^{\rho}B$ for all $\rho < \sigma$. The case when σ is a limit ordinal is trivial, so let $\sigma = \rho + 1$. If $b \in B \cap p^{\sigma}A$ then there is an a_1 in $p^{\rho}A$ such that $pa_1 = b$. Now $\alpha a_1 \in p^{\rho}C[p]$ so *(b')* implies there is a_2 in $p^{\rho}A[p]$ such that $\alpha a_1 = \alpha a_2$. Now $a = a_1 - a_2 \in B \cap p^{\rho}A = p^{\rho}B$ so $b = pa \in p^{\sigma}B$. $\square$

We are now in a position to give conditions corresponding, in our more general setting, to conditions *(a')-(c')*.

3.12 THEOREM. *The following are equivalent for a subgroup B of A (where C denotes the quotient A/B and α the natural epimorphism $A \to C$):*

(a) *B is balanced in A ;*

(b) *to each c in C there is an element a in A such that $\alpha a = c$, $\mathsf{H}(a) = \mathsf{H}(c)$ and $o(a) = o(c)$; and*

(c) *the sequence $0 \to B/B(\mathsf{M}) \to A/A(\mathsf{M}) \to C/C(\mathsf{M}) \to 0$ is exact for all height matrices M .*

Proof. Assuming *(a)* we prove *(b)*. By (3.10), $T(B)$ is balanced in $T(A)$ and (3.9) shows *(b)* holds when $c \in T(C)$. Any a in A satisfying $\mathsf{H}(a) = \mathsf{H}(c)$ and $\alpha a = c$ has infinite order whenever c has infinite order - this proves *(b)*. Conversely, assuming *(b)* we have $T(B)$ balanced in $T(A)$ and (3.11) gives *(a)*. Finally, the commutative diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & B(\mathsf{M}) & \to & A(\mathsf{M}) & \to & C(\mathsf{M}) & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & B & \to & A & \to & C & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & B/B(\mathsf{M}) & \to & A/A(\mathsf{M}) & \to & C/C(\mathsf{M}) & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

with exact columns and exact middle row shows that the first row is exact if and only if the last one is: that is, *(a)* and *(c)* are equivalent. □

Observing that the order of each element in a reduced group is determined by its height matrix, we have the following, perhaps interesting, consequence of *(b)* in (3.12).

3.13 LEMMA. *Let B be a subgroup of a reduced group A. Then B is balanced in A if (and only if) B is H-nice in A.* □

This result does not hold when divisible groups are allowed: in the exact sequence

$$0 \to Z \to Q \to Q/Z \to 0$$

it is clear that Z is H-nice but not isotype in Q.

Having obtained a number of useful characterisations of balanced subgroups, we detail some of their important properties. If $0 \to B \to A \to C \to 0$ is balanced then it is easy to see that the sequence $0 \to D_B \to D_A \to D_C \to 0$ of maximal divisible subgroups is exact, and therefore splits off. On the other hand, every exact sequence of divisible groups is balanced and we are justified in restricting attention to balanced subgroups of reduced groups only.

3.14 PROPOSITION. *If* $0 \to B \to A \to C \to 0$ *is balanced and* $\mathbb{M}$ *is an arbitrary height matrix then* $0 \to B(\mathbb{M}) \to A(\mathbb{M}) \to C(\mathbb{M}) \to 0$ *is balanced.*

Proof. Directly from (2.11). □

The following theorem shows that the balanced sequences form a proper class in the sense of Mac Lane [1].

3.15 THEOREM. *Let* B, C *be subgroups of* A *such that* $C \le B$. *Then:*

(i) if B *is a summand of* A *then* B *is balanced in* A ;

(ii) if C *is balanced in* A *then* C *is balanced in* B ;

(iii) if B *is balanced in* A *then* B/C *is balanced in* A/C ;

(iv) if C *is balanced in* A *and* B/C *in* A/C *then* B *is balanced in* A ;

(v) if C *is balanced in* B *and* B *in* A *then* C *is balanced in* A .

Proof. We may assume A is reduced.

(i) Trivial.

(ii) As C is isotype in B , it suffices to show C is $\mathbb{H}$-nice in B . We do this by proving that if b in $B \backslash C$ is $\mathbb{H}$-proper with respect to C in A , then b is already $\mathbb{H}$-proper with respect to C in B (see (3.6)). Suppose not; by (3.7) there is a prime power p^n and an element c in C such that $h_p^{\,B}(p^n b) < h_p^{\,B}(p^n b+c)$. Now $p^n b$ is p-proper with respect to C in A so that $h_p^{\,A}(p^n b) \ge h_p^{\,A}(p^n b+c)$. Observing that when two elements have different p-heights their sum has the minimum p-height, we have

$$\begin{aligned} h_p^{\ A}(p^n b+c) &\le h_p^{\ A}(c) \\ &= h_p^{\ B}(c) \qquad \text{(as } C \text{ is isotype in } B \text{ and } A\text{)} \\ &= h_p^{\ B}(p^n b) \\ &< h_p^{\ B}(p^n b+c) \text{ , a contradiction.} \end{aligned}$$

(iii) and *(iv)* Directly from (3.13) and (3.8).

(v) Since isotype subgroups have this transitive property, we need only verify H-niceness of C in A. Suppose $a \in A$ and $\mathsf{H}_{A/C}(a+C) = \mathsf{M}$. In the case where $a \in B$ we have $\mathsf{H}_{B/C}(a+C) = \mathsf{M}$ from *(iii)* and so $\mathsf{H}_A(a+c) = \mathsf{M}$ for some c in C. When $a \notin B$ it follows that $\mathsf{H}_{A/B}(a+B) \ge \mathsf{M}$ and $\mathsf{H}_A(a+b) = \mathsf{H}_{A/B}(a+B)$ for some b in B. Now $\mathsf{H}_{A/C}(-a-b+a+C) \ge \mathsf{M}$ so there is c in C such that $\mathsf{H}_B(-b+c) \ge \mathsf{M}$. Therefore $\mathsf{H}_A(a+b-b+c) \ge \mathsf{M}$ and we have $\mathsf{H}_A(a+c) = \mathsf{M}$. □

One consequence of (3.15) is that the balanced extensions of a group B by a group C form a subgroup of $\operatorname{Ext}(C, B)$.

3.16 DEFINITION. Let B and C be groups. Then we denote by $\operatorname{Bext}(C, A)$ that subgroup of $\operatorname{Ext}(C, B)$ consisting of the balanced extensions of B by C. □

As we have seen, many of the well known properties of balanced subgroups of torsion groups carry over directly to general balanced subgroups. We list some more, with short proofs where necessary.

(A) *If B is balanced in A then*

$$f_\sigma^p(A) = f_\sigma^p(A/B) + f_\sigma^p(B)$$

for all primes p and ordinals σ.

It follows from (3.10) and *(b')* that the sequence

$$0 \to p^\sigma B[p] \to p^\sigma A[p] \to p^\sigma C[p] \to 0$$

is exact for all primes p and ordinals σ .

(B) *Let* $\{A_i : i \in I\}$ *be a set of groups and let* B_i *be a subgroup of* A_i *for each* i *in* I . *Then* $\bigoplus_{i\in I} B_i$ *is balanced in* $\bigoplus_{i\in I} A_i$ *if and only if* B_i *is balanced in* A_i *for every* i *in* I .

(C) *Every subgroup of* A *is balanced if and only if* A *is elementary torsion.*

Since it is impossible for every subgroup of a group with elements of infinite order to be isotype, A must be torsion. For every non-zero a in A_p , the bounded subgroup $\langle pa \rangle$ is pure in A and therefore in $\langle a \rangle$. Thus $\langle pa \rangle$ is a summand of $\langle a \rangle$ which only happens when $pa = 0$.

(D) *If* B *is balanced in* A *then* $\operatorname{Tor}(B, X)$ *is balanced in* $\operatorname{Tor}(A, X)$ *for every group* X .

For each prime p and ordinal σ , application of the functor $\operatorname{Tor}(-, p^\sigma X)$ to the pure exact sequence

$$0 \to p^\sigma B \to p^\sigma A \to p^\sigma C \to 0$$

yields the pure exact sequence

$$0 \to \operatorname{Tor}(p^\sigma B, p^\sigma X) \to \operatorname{Tor}(p^\sigma A, p^\sigma X) \to \operatorname{Tor}(p^\sigma C, p^\sigma X) \to 0$$

(see Fuchs [1]). Nunke [2] has shown that

$$p^\sigma \operatorname{Tor}(A, X) = \operatorname{Tor}(p^\sigma A, p^\sigma X)$$

for every prime p and ordinal σ , so we have

$$0 \to p^\sigma \operatorname{Tor}(B, X) \to p^\sigma \operatorname{Tor}(A, X) \to p^\sigma \operatorname{Tor}(C, X) \to 0 .$$

Since $\operatorname{Tor}(A, X)$ is torsion for each A and X it follows that $\operatorname{Tor}(B, X)$ is balanced in $\operatorname{Tor}(A, X)$.

(E) *If* B *is a pure subgroup of* A *with* A/B *torsion and* $p^{\omega}(A/B) = 0$ *for all primes* p *then* B *is balanced in* A .

For $\mathcal{H}$-niceness of B in A we need only consider non-zero elements x in $(A/B)_p$ for each prime p separately. The p-height of such an x is finite, so there is clearly an element of maximal p-height in the coset x . Purity of B in A yields $p^n B = B \cap p^n A$ when $n < \omega$ while the condition on C shows that $p^{\sigma} B = p^{\sigma} A$ for all $\sigma \geq \omega$ so B is isotype in A .

We conclude with the remark that, with appropriate modification, all the results of this chapter except (3.13) and the last statement of (3.10) also hold for $\mathcal{H}$-balanced subgroups.

CHAPTER 4

BALANCED INJECTIVES

In this chapter we show that the balanced injectives are just the pure injectives (otherwise known as the algebraically compact groups of Kaplansky [1]). These groups are well known, and are in fact characterised by isomorphism invariants (see Fuchs [1]). Thus we have a very complete description of the balanced injectives.

Our argument in (4.3) extends that given by Griffith [2] who characterised the balanced injectives in the category of all p-groups. A consequence of this more general argument is that the injectives with respect to balanced exact sequences of torsion free groups are just the algebraically compact groups of Harrison [1].

Recall that A is cotorsion if A is a summand in every containing group G with G/A torsion free (since G/A torsion free implies that A is pure in G, it is clear that every pure injective group is cotorsion, but the converse is not true (see Fuchs [1])). The following well known homological characterisations will be of use to us. Let $\mathrm{Pext}(C, A)$ be the subgroup of $\mathrm{Ext}(C, A)$ consisting of all pure extensions of A by C. Then A is cotorsion if and only if $\mathrm{Ext}(Q, A) = 0$, and A is pure injective if and only if $\mathrm{Pext}(Q \oplus Q/Z, A) = 0$.

We give two technical lemmas required for the proof of our main theorem.

4.1 LEMMA. *Let C be torsion free and homogeneous of type $(0, \ldots, 0, \ldots)$. Then every short exact sequence*

$$0 \to B \to A \xrightarrow{\eta} C \to 0$$

is balanced.

Proof. Suppose $c \in C$. Then there is an element c' in C and a positive integer n such that $nc' = c$ and $H(c') = \langle\langle 0, \ldots, 0, \ldots\rangle\rangle$.

Any element a' in A for which $\eta a' = c'$ also satisfies $\eta n a' = c$ and $\mathrm{H}(na') = \mathrm{H}(c)$. $\square$

Let $p_1, p_2, \ldots$ denote the sequence of all primes in the natural order.

4.2 LEMMA. *Let the groups* A_k *for* $k = 0, 1, \ldots$ *be such that*

$$A_0 \cong Z \quad \text{and} \quad A_k \cong \bigoplus_{0<r<\omega} Z\left(p_k^{\,r}\right) \quad \text{when} \quad k \geq 1 \ .$$

For a given infinite cardinal $\underline{\underline{m}}$ *let*

$$B_k = \prod_{\underline{\underline{m}}} A_k \quad \text{and} \quad C_k = \bigoplus_{\underline{\underline{m}}} A_k \quad \text{for} \quad k = 0, 1, \ldots \ .$$

Then there is a subgroup G_k *of* B_k *for* $n = 0, 1, \ldots$ *satisfying*

$$G_0/C_0 \cong \bigoplus_{\underline{\underline{m}}^{\aleph_0}} Q$$

and

$$G_k/C_k \cong \bigoplus_{\underline{\underline{m}}^{\aleph_0}} Z\left(p_k^{\,\infty}\right) \quad \text{when} \quad k \geq 1 \ .$$

Proof. It suffices to demonstrate that the divisible part of B_0/C_0 and $\left(B_k/C_k\right)_{p_k}$ for $k \geq 1$ has cardinality at least $\underline{\underline{m}}^{\aleph_0}$. In preparation for this we first show that each set I of cardinality $\underline{\underline{m}}$ has a family of $\underline{\underline{m}}^{\aleph_0}$ subsets of cardinality $\aleph_0$, any two of which have finite intersection.

Let X^Y denote the set of maps of Y into X and n the set $\{1, 2, \ldots, n\}$ of the first n positive integers. Writing $S = \bigcup_{n<\omega} I^{\mathrm{n}}$ and $N = \bigcup_{n<\omega} \mathrm{n}$ we have $|S| = |I|$ and $\left|I^N\right| = \underline{\underline{m}}^{\aleph_0}$. Thus there is a bijection $S \xrightarrow{\sigma} I$. Denote the restriction of a map f in I^N to n by $f|_{\mathrm{n}}$ and the subset $\left\{\sigma\left(f|_{\mathrm{n}}\right) : n = 1, 2, \ldots\right\}$ of I by $\overline{f}$. We claim that the $\bar{f}$ are the desired subsets of I. To see this, first note that $|\bar{f}| = \aleph_0$ for each f in I^N. Suppose f and g are distinct maps in I^N; say, $f(k) \neq g(k)$ for some k in N. If $i \in \bar{f} \cap \bar{g}$ then $i = \sigma(f|_m) = \sigma(g|_n)$ for suitable m and n in N; as σ is monic, $f|_m = g|_n$ which is impossible unless $m = n < k$. Thus $|\bar{f} \cap \bar{g}| < k$. As distinct maps f in I^N yield distinct subsets $\bar{f}$ of I, we must have $|\{\bar{f} : f \in I^N\}| = \underline{\underline{m}}^{\aleph_0}$.

Now put $B_k = \prod_{i \in I} A_k$ and when $k \geq 1$ define, for each f in I^N, an element y_f in B_k by choosing its component in the i-th copy of A_k as 0 if $i \notin \bar{f}$ and as any element of order p_k and p_k-height n when $i = \sigma(f|_n)$. Observe that for each n, almost all of the entries of y_f have p_k-height greater than or equal to n. Thus we obtain a sequence $x_1, x_2, \ldots$ such that $p_k(x_1 + C_k) = y_f + C_k$ and $p_k(x_{r+1} + C) = x_r + C$ and such that, for each n, almost all of the entries of x_r have p_k-height greater than or equal to n. It follows that each $y_f + C$ has p_k-height ∞ and it is a routine matter to see that $\{y_f + C_k : f \in I^N\}$ is an independent set of $\underline{\underline{m}}^{\aleph_0}$ elements. The case $k = 0$ is proved similarly. □

Now for the promised characterisation of balanced injectives.

4.3 THEOREM. *A group A is balanced injective if and only if A is pure injective.*

Proof. Let A be balanced injective. We use a counting argument to show A is pure injective. Choose a cardinal $\underline{\underline{m}}$ such that $|A| \leq \underline{\underline{m}}$ and such that $\underline{\underline{m}}^{\aleph_0} > \underline{\underline{m}}$; for instance

$$\underline{\underline{m}} = |A| + 2^{|A|} + 2^{2^{|A|}} + \dots$$

is such a cardinal (see Griffith [2] for details of one proof). For the rules of exponentiation of cardinals used here, see Kuratowski and Mostowski [1], especially p. 287. We remark that since $2^{\underline{\underline{n}}} = \underline{\underline{n}}^{\underline{\underline{n}}}$ for every infinite cardinal $\underline{\underline{n}}$, we have $\underline{\underline{n}} \leq \underline{\underline{n}}^{\aleph_0} \leq 2^{\underline{\underline{n}}}$. For each $n = 0, 1, \dots$ let

$$0 \to U_n \xrightarrow{\phi_n} V_n \xrightarrow{\psi_n} G_n \to 0 \tag{12}$$

be a pure projective resolution of the group G_n given in (4.2). A glance at (4.1) when $n = 0$ and (E) on p. 42 when $n > 0$ reveals that (12) is balanced for every n. Write $U = \bigoplus_{n<\omega} U_n$, $V = \bigoplus_{n<\omega} V_n$, $G = \bigoplus_{n<\omega} G_n$,

$\phi = \bigoplus_{n<\omega} \phi_n$ and $\psi = \bigoplus_{n<\omega} \psi_n$. The corresponding exact sequence

$0 \to U \xrightarrow{\phi} V \xrightarrow{\psi} G \to 0$ is balanced and induces the exact sequence

$$\dots \operatorname{Hom}(V, A) \xrightarrow{\phi^*} \operatorname{Hom}(U, A) \to \operatorname{Pext}(G, A) \to \operatorname{Pext}(V, A) = 0$$

where the last term is zero since V is pure projective and ϕ^* is an epimorphism since A is balanced injective. Thus we get $\operatorname{Pext}(G, A) = 0$. Now the pure exact sequence

$$0 \to \bigoplus_{n<\omega} C_n \to \bigoplus_{n<\omega} G_n = G \to \bigoplus_{n<\omega} D_n \to 0$$

where $D_n = G_n/C_n$ yields the exact sequence

$$\dots \operatorname{Hom}\left(\bigoplus_{n<\omega} C_n, A\right) \xrightarrow{f} \operatorname{Pext}\left(\bigoplus_{n<\omega} D_n, A\right) \to \operatorname{Pext}(G, A) = 0$$

so that f is epic. For convenience of expression we write $Z(p_0^\infty)$ in place of Q. Then for $n = 0, 1, \dots$ it follows that $D_n = \bigoplus_{\underline{\underline{m}}^{\aleph_0}} Z(p_n^\infty)$.

If $\operatorname{Pext}(Q \oplus Q/Z, A) \neq 0$ then

$$\left|\operatorname{Pext}\left(\bigoplus_{n<\omega} D_n, A\right)\right| = \prod_{n<\omega} \prod_{\underline{\underline{m}}^{\aleph_0}} |\operatorname{Pext}(Z(p_n^\infty), A)| \geq 2^{\underline{\underline{m}}^{\aleph_0}} .$$

But

$$\left|\operatorname{Hom}\left(\bigoplus_{n<\omega} C_n,\ A\right)\right| = \prod_{n<\omega} |\operatorname{Hom}(C_n,\ A)| \le \prod_{n<\omega}\prod_{\underline{\underline{m}}} |A| \le \underline{\underline{m}}^{\underline{\underline{m}}} = 2^{\underline{\underline{m}}}\ .$$

Now $2^{\underline{\underline{m}}^{\aleph_0}} > 2^{\underline{\underline{m}}}$ which contradicts the fact that f is epic. Therefore $\operatorname{Pext}(Q \oplus Q/Z,\ A) = 0$.

The converse follows immediately from the observation that balanced subgroups are pure. □

4.4 COROLLARY (to proof of 4.3). *A group A is injective with respect to balanced exact sequences of torsion free groups if and only if A is cotorsion.*

Proof. Apply the argument of (4.3) with G_0 in place of G . □

Let B be a subgroup of A . C.L. Walker [2] defines B to be *regular* in A if for every x in A/B and every rank 1 subgroup C/B of A/B containing x there is a c in the coset x such that $o(x) = o(c)$ and $\mathrm{H}_C(c) = \mathrm{H}_{C/B}(x)$, and then asks for a description of injectives with respect to regular exact sequences. Since regularity lies between purity and balanced, (4.3) yields an immediate answer to this question.

4.5 COROLLARY. *A group A is injective with respect to regular exact sequences if and only if A is pure injective.* □

A natural question that arises is: can one imbed every group as a balanced subgroup of a balanced injective? The answer is no. This is clear from the following proposition.

4.6 PROPOSITION. *Every balanced subgroup of a balanced injective is again balanced injective.*

Proof. Let B be a balanced subgroup of a pure injective group A .

We may assume A reduced. Now $\bigcap_{n<\omega} n!A = 0$ so $\bigcap_{n<\omega} n!(A/B) = 0$ and Corollary 39.2 of Fuchs [1] shows that B is also pure injective. As B is pure in A, we also have B a summand of A. □

REMARKS. 1. (4.6) should be contrasted with (6.3) which asserts that every group is a balanced homomorphic image of a balanced projective.

2. The results (4.1)-(4.4) hold for H-balanced exact sequences. In particular A is balanced injective if and only if A is H-balanced injective.

CHAPTER 5

GROUPS WITH TORSION FREE RANK 1

In this chapter we study two classes of groups (denoted by $\mathcal{A}$ and $\mathcal{C}$) whose members have torsion free rank at most 1. Groups in $\mathcal{A}$ are the 'building blocks' for our study of the balanced projectives, and groups in $\mathcal{C}$ are similarly fundamental to our work with H-projectives. As might be expected, $\mathcal{C}$ is a subclass of $\mathcal{A}$ and each class contains the totally projective groups. Many of the results that are required for groups in these two classes also apply to arbitrary groups with torsion free rank 1. Accordingly we first examine a general theory of groups with torsion free rank 1. Our approach is to consider such groups as extensions; if A has torsion free rank 1 and a is an element in A with infinite order, then A is an extension

$$0 \to \langle a \rangle \to A \xrightarrow{\eta} T^* \to 0 \tag{13}$$

of the free group $\langle a \rangle$ by the torsion group T^*. With each extension (13) we associate the pair $(\mathbb{M}, T^*)$ where $\mathbb{M} = \mathbb{H}(a)$. Necessary conditions are given for an arbitrary pair $(\mathbb{M}, T^*)$ to be associated with an exact sequence (13) and these conditions are shown to be sufficient when T^* is totally projective. When T^* is totally projective, the pair $(\mathbb{M}, T^*)$ is shown to determine A up to isomorphism, thus yielding a complete description of all possible extensions (13) in this special case. The conditions on a pair $(\mathbb{M}, T^*)$ turn out to involve only the entries of $\mathbb{M}$ and the Ulm invariants of T^*. This is perhaps surprising when contrasted with the conditions for a pair $(\mathbb{M}, T)$ to be associated with an exact sequence (13) such that $\mathbb{M} = \mathbb{H}(a)$ and $T = T(A)$ (see p. 54).

The asterisk is used throughout to indicate that a group is being considered, in some sense, as the last term of a sequence like (13).

5.1 DEFINITION. Let $\mathbb{M}$ be a height matrix, $\mathfrak{u}$ a p-indicator and

T^* a torsion group. The pair (M, T^*) (the pair (u, T^*)) is said to be *admissible* if there is a group A of torsion free rank 1 containing an element a of infinite order such that $\mathsf{H}(a) = \mathsf{M}$ $(U_p(a) = \mathsf{u})$ and $A/\langle a\rangle \cong T^*$. □

The problem of finding admissible pairs is 'localised' to a single prime in our first result.

5.2 PROPOSITION. *Let* M *be a height matrix,* T^* *a torsion group. Then* (M, T^*) *is admissible if and only if the pairs* (M_p, T^*_p) *are admissible for every prime* p *.*

Proof. Suppose a is an element of infinite order in a group A of torsion free rank 1 such that $\mathsf{H}(a) = \mathsf{M}$ and $A/\langle a\rangle \cong T^*$. Let $A_{(p)}$ be the complete inverse image of T^*_p in A ; then $A/A_{(p)}$ is torsion with no p-component so $A_{(p)}$ is p-isotype in A and $U_p^{A_{(p)}}(a) = U_p^A(a) = \mathsf{M}_p$.

Conversely suppose we are given, for each prime p , a group $A_{(p)}$ containing an element a_p of infinite order such that $U_p(a_p) = \mathsf{M}_p$ and $A_{(p)}/\langle a_p\rangle \cong T^*_p$. Since $A_{(p)}/\langle a_p\rangle$ has no q-torsion it is evident from (2.1) that $U_q(a_p) = (0, 1, 2, \ldots)$ for every $q \neq p$. Write $B = \bigoplus_p A_{(p)}$ and let C be the subgroup of B generated by $\{a_p - a_q : p, q \in P\}$. We claim that if $A = B/C$ then the element $a = a_p + C$ $(= a_q + C$ for every other prime $q)$ is such that $A/\langle a\rangle \cong T^*$ and $\mathsf{H}(a) = \mathsf{M}$. To see this, note that $\langle a_p, C\rangle = \bigoplus_{q\in P} \langle a_q\rangle$ so $A/\langle a\rangle \cong B/\langle a_p, C\rangle \cong T^*$ while $a_p + C$ lies in the isomorphic copy $(A_{(p)}+C)/C$ of $A_{(p)}$ which is p-isotype in A (because the factor group has no p-torsion) so

$$U_p^{A_{(p)}}(a_p) = U_p^A(a_p+C) = U_p^A(a) . \qquad \square$$

Fundamental to our treatment of a group A of torsion free rank 1 as an extension (13) is the observation that η imbeds $T(A)$ as a 'large' subgroup of T^*.

5.3 LEMMA. *Let* a *be an element of infinite order in a group* A *of torsion free rank* 1, *and* $\eta : A \to A/\langle a \rangle = T^*$ *the natural epimorphism. Then the factor group* $T^*/\eta T(A)$ *is locally cyclic.*

Proof. We have

$$T^*/\eta T(A) = A/\langle a\rangle / \big(T(A)+\langle a\rangle\big)/\langle a\rangle$$
$$\cong A/\big(T(A)+\langle a\rangle\big)$$

so that $T^*/\eta T(A)$ is a homomorphic image of $A/T(A)$. Since $A/T(A)$ is locally cyclic, so is $T^*/\eta T(A)$. □

Thus the relationship between $T(A) \cong \eta T(A)$ and T^* in (13) is indeed a close one. In fact, the Ulm invariants of $T(A)$ (which are the same as those of A) and T^* can be related *via* $\mathsf{H}(a)$; we do this in the next three lemmas. The restriction of $T^* = A/\langle a\rangle$ to a p-group in the proofs of these lemmas affords no loss of generality - if $A_{(p)}$ is the group derived from A in (5.2) then $T\big(A_{(p)}\big) = A_p$ and

$$U_p^{A_{(p)}}(a) = U_p^{A}(a)$$

and

$$f_\sigma^{\,p}\big(A_{(p)}\big) = f_\sigma^{\,p}(A) \quad \text{for } \sigma \text{ an ordinal or } \infty .$$

Thus we can use $A_{(p)}$ and M_p for each prime p separately to find the desired relationships.

5.4 LEMMA. *Let* A *have torsion free rank* 1 *and suppose* $T(A)$ *is reduced. Then the divisible part of* $(A/\langle a\rangle)_p$ *is* $Z(p^\infty)$ *if* $U_p(a)$ *contains* ∞, *and* 0 *otherwise.*

Proof. Assume that $A/\langle a \rangle$ is a p-group and let $\eta : A \to A/\langle a \rangle$ be the natural homomorphism. It follows from (2.24) that $p^{\infty}\eta A = \eta p^{\infty} A$. Thus ηA is reduced unless $p^{\infty} A \neq 0$. Since $p^{\infty} T(A) = 0$, we have $p^{\infty} A \neq 0$ exactly when $U_p(a)$ contains ∞ . That the divisible part of ηA cannot exceed $Z(p^{\infty})$ is evident from the fact that $p^{\infty} A \cap T(A) = 0$ implies $p^{\infty} A$ is torsion free of rank at most 1 . □

5.5 LEMMA. *Let A have torsion free rank 1 and suppose a is an element of A having infinite order and p-indicator $(\sigma_0, \sigma_1, \ldots)$. Then for every ordinal σ we have*

$$f_{\sigma}^{p}(A/\langle a \rangle) = \begin{cases} f_{\sigma}^{p}(A, \langle a \rangle)+1 & \text{if } \sigma+1 = \sigma_i \text{ and a gap precedes } \sigma_i \text{ ; and} \\ f_{\sigma}^{p}(A, \langle a \rangle) & \text{otherwise.} \end{cases}$$

Proof. Assume $A/\langle a \rangle$ is a p-group. Let $\bar{B}$ $(\bar{b})$ denote the image of a subset B (element b) under the natural map $A \to A/\langle a \rangle$. It follows from (2.24) that $p^{\sigma}\bar{A} = \overline{p^{\sigma} A}$, while it is immediate from (5.3) that the dimension of $\overline{p^{\sigma} A}[p]$ (as a Z/pZ vector space) is at most one greater than the dimension of its subspace $\overline{p^{\sigma} A[p]}$. It follows that

$$\begin{aligned} f_{\sigma}^{p}(A, \langle a \rangle) &= \dim\{p^{\sigma}A[p]/\left((p^{\sigma+1}A+\langle a \rangle) \cap p^{\sigma}A[p]\right)\} \\ &= \dim\{\left(p^{\sigma}A[p]+p^{\sigma+1}A+\langle a \rangle\right)/\left(p^{\sigma+1}A+\langle a \rangle\right)\} \\ &= \dim\{\left(\overline{p^{\sigma}A[p]}+p^{\sigma+1}\bar{A}\right)/p^{\sigma+1}\bar{A}\} \\ &\leq \dim\{\left(p^{\sigma}\bar{A}[p]+p^{\sigma+1}\bar{A}\right)/p^{\sigma+1}\bar{A}\} \\ &= \dim\{p^{\sigma}\bar{A}[p]/p^{\sigma+1}\bar{A}[p]\} \\ &= f_{\sigma}^{p}(A/\langle a \rangle) \\ &\leq f_{\sigma}^{p}(A, \langle a \rangle) + 1 \end{aligned}$$

and therefore $f_\sigma^p(A, \langle a\rangle) < f_\sigma^p(A/\langle a\rangle)$ if and only if

$$\overline{p^\sigma A[p]} + p^{\sigma+1}\overline{A} < p^\sigma\overline{A}[p] + p^{\sigma+1}\overline{A} .$$

Suppose σ is indeed such that $f_\sigma^p(A, \langle a\rangle) < f_\sigma^p(A/\langle a\rangle)$. Then p-niceness of $\langle a\rangle$ in A ensures the existence of a b in A satisfying $h_p(b) = \sigma$ and

$$\overline{b} \in p^\sigma\overline{A}[p]\setminus\left(p^{\sigma+1}\overline{A}+\overline{p^\sigma A[p]}\right) .$$

We have $pb = ra$ for some integer r . If $\sigma + 1 = h_p(b) + 1 < h_p(ra)$ then there exists c in $p^{\sigma+1}A$ with $pc = ra = pb$, so that $b - c \in p^\sigma A[p]$, whence $\overline{b} \in p^{\sigma+1}\overline{A} + \overline{p^\sigma A[p]}$, a contradiction. Thus $\sigma + 1 = h_p(ra) = \sigma_i$ for some i . If $\sigma = \sigma_{i-1}$ then

$$h_p(b) = \sigma = \sigma_{i-1} = h_p(r'a) ,$$

where $pr' = r$ and $pb = p(r'a)$, so that $b = r'a + t$ such that $t \in p^\sigma A[p]$. Then $\overline{b} = \overline{t} \in \overline{p^\sigma A[p]}$, a contradiction. Thus $\sigma \neq \sigma_{i-1}$ and since $\sigma + 1 = \sigma_i$ it follows that a gap precedes σ_i .

Conversely, suppose $\sigma + 1 = \sigma_i$ for some i and a gap precedes σ_i . Then $h_p\left(p^i a\right) = \sigma + 1$ and $p^i a = pb$ with $h_p(b) = \sigma$. For this b it follows that $\overline{b} \notin p^{\sigma+1}\overline{A} + \overline{p^\sigma A[p]}$; if not, the p-niceness of a in A yields $b = c + x + sa$ such that s is an integer, $x \in p^\sigma A[p]$ and $c \in p^{\sigma+1}A$. This implies $h_p(sa) \geq \sigma$ and $h_p(psa) = \sigma + 1$ so in fact $h_p(sa) = \sigma$, contradicting $\sigma \neq \sigma_j$ for $j < i$. □

5.6 LEMMA. *Let* A *have torsion free rank* 1 *and suppose* a *is an element of* A *having infinite order and p-indicator* $(\sigma_0, \sigma_1, \ldots)$. *Then for* σ *an ordinal or* ∞ *we have*

$$f_\sigma^{\,p}(A) = \begin{cases} f_\sigma^{\,p}(A/\langle a\rangle)+1 & \textit{if } \sigma = \sigma_n \textit{ and a gap follows } \sigma_n\,; \\ f_\sigma^{\,p}(A/\langle a\rangle)-1 & \textit{if there is an } n \textit{ such that } \sigma+1 = \sigma_n \textit{ and a gap precedes } \sigma_n\,; \\ f_\sigma^{\,p}(A/\langle a\rangle) & \textit{otherwise.} \end{cases}$$

Proof. Assume $A/\langle a\rangle$ is a p-group. Wallace [1] has shown that when σ is an ordinal

$$f_\sigma^{\,p}(A, \langle a\rangle) = \begin{cases} f_\sigma^{\,p}(A)-1 & \text{if } \sigma = \sigma_n \text{ and a gap follows } \sigma_n\,; \text{ and} \\ f_\sigma^{\,p}(A) & \text{otherwise.} \end{cases}$$

The case for σ an ordinal now follows from (5.5) while the case when $\sigma = \infty$ is immediate from (5.4). $\square$

If A has torsion free rank 1 and a in A has infinite order, we have shown in (5.6) that the Ulm invariants of A (and therefore of $T(A)$) are uniquely determined by those of the quotient $A/\langle a\rangle$ in the presence of $\mathsf{H}(a)$ and vice versa. With this in mind we review what is known about the connection between $\mathsf{H}(a)$ and $T(A)$. Let $\mathsf{H}(a) = [\sigma_{pk}]$ and write $T = T(A)$. Then the following three conditions must be satisfied by $\mathsf{H}(a)$ and T (see Fuchs [2], p. 200-201):

(i) *if there is a gap following* σ_{pk} *then* $f_{\sigma_{pk}}^{\,p}(T) \neq 0$;

(ii) *if* $\sigma_{pk} \neq \infty$ *for all* k *then*

$$\sigma_{pk} < l_p(T) + \omega\,,$$

while if $\sigma_{pr} = \infty$ *for some* r *then* $\sigma_{pk} < l_p(T)$ *whenever* $\sigma_{pk} \neq \infty$; *and*

(iii) *for every prime* p *for which there is an integer* $m \geq 0$ *such that* $p^{\omega\sigma+m}T_p = 0$ *and* $p^{\omega(\sigma-1)}T_p$ *is torsion complete or*

$T_p/p^{\omega\sigma}T_p$ is the torsion part of the inverse limit

$$\varprojlim T_p/p^{\rho}T_p \quad (\rho \to \omega\sigma)$$

(according as σ is an isolated or limit ordinal), then $\sigma_{pk} \neq \infty$ implies

$$\sigma_{pk} < \max\big(l_p(T_p), \omega\big) .$$

In fact, these conditions are also sufficient:

5.7 THEOREM (Fuchs [2]). *Let $\mathsf{M} = [\sigma_{pk}]$ be a height matrix, and let T be a reduced torsion group. Then there is a mixed group A of torsion free rank 1 whose torsion part is T and which contains an element a of infinite order such that*

$$\mathsf{H}(a) = \mathsf{M}$$

if and only if M and T satisfy (i)-(iii). □

It is natural to ask the corresponding question for pairs (M, T^*), namely: which pairs (M, T^*) are admissible? This question would be of little interest if (5.7) were to provide an immediate answer; that is, if the strong connection between M, $T(A)$ and T^* already revealed by (5.6) and (5.3) would in fact amount to T^* being determined up to isomorphism by M and $T(A)$. However, the following example due to Megibben [1] shows that neither T^* nor A is determined up to isomorphism by M and $T(A)$.

5.8 EXAMPLE (Megibben [1]). Let B be an unbounded countable direct sum of cyclic p-groups and $\overline{B}$ the torsion completion of B (the torsion completion is described on p. 15, Fuchs [2]). Choosing a pure subgroup T of $\overline{B}$ such that $B \leq T$ and $\overline{B}/T \cong Z(p^{\infty})$, we let H be a subgroup of $\mathrm{Ext}\big(Z(p^{\infty}), T\big)$ such that $T \leq H$, $H/T \cong Q$, and $\bigcap_{n=1}^{\infty} n!H = H^1 \neq 0$.

Similarly we find a subgroup G of $\mathrm{Ext}\big(Z(p^{\infty}), B\big)$ such that $B \leq G$, $G/B \cong Q$, and $G^1 \neq 0$ so that there are elements g in G and h in H

of infinite order such that $h_p(p^k g) = \omega + k = h_p(p^k h)$ for $k = 0, 1, \ldots$. Writing $A = G \oplus T$ and $C = H \oplus B$ we have $\mathsf{H}_A(g) = \mathsf{H}_C(h)$ and $T(A) = B \oplus T \cong T(C)$. However, since

$H/H^1 \cong \overline{B}$, $G/G^1 \cong B$ and $B \oplus T \ncong \overline{B} \oplus B$ (see Fuchs [2], p. 206) we have

$$(A/\langle g\rangle)_p \cong B \oplus T \ncong B \oplus \overline{B} \cong (C/\langle h\rangle)_p \ . \qquad \square$$

Conversely, we might ask whether M and T^* determine $T(A)$ up to isomorphism. This question is immediately answered in the negative with the following example.

5.9 EXAMPLE. Let B, C, G, T, g and h be the same as in (5.8). Since $\mathsf{H}(g) = \mathsf{H}(h) = \mathsf{M}$ has all entries ∞ except those in the row corresponding to p (this row has the form $(\omega, \omega{+}1, \ldots)$), we deduce from (5.6) that the Ulm invariants of $B = T(G)$, and $(G/\langle g\rangle)_p$ are the same; these two groups are countable and reduced so $B \cong (G/\langle g\rangle)_p$. Thus if $F = \overline{B} \oplus G$, $D = \bigoplus_{q \neq p} Z(q^\infty)$ and $T^* = \overline{B} \oplus B \oplus D$ then F and C satisfy $F/\langle g\rangle \cong T^*$ and $C/\langle h\rangle \cong T^*$ respectively, while

$$T(F) \cong \overline{B} \oplus B \ncong T \oplus B \cong T(C)$$

and $\mathsf{H}_F(g) = \mathsf{H}_C(h)$. $\square$

Thus (5.6) and (5.7) do not yield a description of those pairs (M, T^*) which are admissible; the reason is, of course, that the Ulm invariants do not serve to determine a torsion group up to isomorphism. Let us therefore take a closer look at the way in which η (in (13)) imbeds $T(A)$ in T^* .

5.10 LEMMA. *Let A be reduced and a an element of infinite order in A such that $A/\langle a\rangle = T^*$ is a p-group. Let $\eta : A \to T^*$ be the natural homomorphism and set $T = T(A)$, $\sigma = h_p(a)$ and let λ be a limit ordinal such that $\lambda \leq \sigma$. Then*

1. *ηT is λ-dense in T^* ;*

2. $p^{\rho}\eta T = \eta T \cap p^{\rho}T^*$ *whenever* $\rho \leq \sigma$; *and*

3. $T^*/p^{\rho}T^* \cong T/p^{\rho}T$ *whenever* $\rho < \lambda$.

Furthermore, when T^* *is reduced there is an integer* $k \geq 0$ *such that*

$$l_p(T) + k = l_p(T^*)$$

and when T^* *is not reduced there is an integer* $k \geq 0$ *such that*

$$l_p(T^*) + k = l_p(T) .$$

In particular, if $U_p(a)$ *has infinitely many gaps then* $l_p(T^*) = l_p(T)$.

Finally if $\rho = \sup_{k<\omega} h_p(p^k a)$ *then*

$$p^{\rho}\eta T = p^{\rho}T^* .$$

Proof. 1. The subgroup ηT of T^* is λ-dense in T^* if for each t^* in T^* and each ordinal ρ such that $\rho < \lambda$ we have $t^* \in \eta T + p^{\rho}T^*$. Since η is epic there is x in A such that $\eta x = t^*$. When $x \in T$ the required inclusion of t^* in $\eta T + p^{\rho}T^*$ is trivial, so we need only deal with the case $o(x) = \infty$. Let n and k be integers such that $0 \neq p^n x = ka \in \langle a \rangle$. The condition on ρ implies that $\rho + \omega \leq \lambda \leq \sigma$, so there is a strictly increasing finite sequence $\rho_0, \rho_1, \ldots, \rho_n$ of ordinals such that $\rho_0 \geq \rho$ and $\rho_n = h_p(ka)$. As we observed on p. 13 in chapter 2, there is a y in A such that $h_p(y) \geq \rho_0$ and $p^n y = p^n x$. Now $x = y + t$, where $t \in T$ so

$$t^* = \eta x = \eta y + \eta t \in p^{\rho}T^* + \eta T .$$

2. Let ρ be an ordinal such that $\rho \leq \sigma$. It suffices to prove that $p^{\rho}\eta T \geq \eta T \cap p^{\rho}T^*$. If $x \in \eta T \cap p^{\rho}T^*$ then p-niceness of $\langle a \rangle$ in A (see (2.24)) yields $x = \eta y = \eta t$ such that $y \in p^{\rho}A$ and $t \in T$. Thus $y - t \in \langle a \rangle \leq p^{\rho}A$ which implies that $t \in p^{\rho}T$ and $x = \eta t \in \eta p^{\rho}T \leq p^{\rho}\eta T$.

3. From 1 and 2 we have

$$T^*/p^\rho T^* = \left(p^\rho T^*+\eta T\right)/p^\rho T^*$$
$$\cong \eta T/p^\rho \eta T$$
$$\cong T/p^\rho T$$

for every ordinal ρ such that $\rho < \lambda$.

For the next statement of the lemma, suppose T^* is reduced. Then $T \cong \eta T \leq T^*$ implies $l_p(T) \leq l_p(T^*)$, while (5.4) shows $h_p\left(p^k a\right) = \sigma_k < \infty$ for $k = 0, 1, \ldots$. We consider the sequence $\sigma_0, \sigma_1, \ldots$ and distinguish two cases.

Case I. The sequence $\left(\sigma_0, \sigma_1, \ldots\right)$ has only a finite number of gaps. If σ_n is the greatest term with a gap preceding it (recall that a gap always precedes σ_0) then let $\sigma_n = \gamma + r$ with γ zero or a limit ordinal and r finite. Condition *(ii)* of (5.7) implies that $l_p(T) \geq \gamma$ and (5.6) shows $f_\rho^{\,p}(T) = f_\rho^{\,p}(T^*)$ whenever $\rho \geq \sigma_n$. Therefore $l_p(T) + r \geq l_p(T^*) \geq l_p(T)$.

Case II. The sequence $\left(\sigma_0, \sigma_1, \ldots\right)$ has infinitely many gaps. If $\gamma = \sup_i \sigma_i$ then condition *(i)* of (5.7) implies $l_p(T) \geq \gamma$. Since $f_\rho^{\,p}(T) = f_\rho^{\,p}(T^*)$ when $\rho \geq \gamma$, it follows that $l_p(T) = l_p(T^*)$.

Suppose T^* is not reduced; then T^* has a (unique) summand $Z(p^\infty)$. Now

$$T^*/\eta T \cong Z(p^\infty) \cong \left(Z(p^\infty)+\eta T\right)/\eta T$$

implies $Z(p^\infty) + \eta T = T^*$. Writing $T^* = Z(p^\infty) \oplus B$ we have

$$B \cong T^*/Z(p^\infty) = \left(\eta T+Z(p^\infty)\right)/Z(p^\infty)$$
$$\cong \eta T/\left(\eta T \cap Z(p^\infty)\right)$$
$$= \eta T/C$$

where C is cyclic because ηT is reduced. Let $\gamma = l_p(B) = l_p(T^*)$. As C is finite and therefore p-nice in ηT it follows that $l_p(\eta T) \geq \gamma$ and

$p^{\gamma}\eta T \le C$. Thus $l_p(T^*) \le l_p(\eta T) \le l_p(T^*) + k$ for some integer $k \ge 0$.

It is clear from the above that $l_p(T) \ne l_p(T^*)$ can only occur when the sequence $(\sigma_0, \sigma_1, \ldots)$ has finitely many gaps. Finally, if $\rho = \sup_i \sigma_i$ and $p^{\rho}\eta T \ne p^{\rho}T^*$ let x in $p^{\rho}A$ satisfy $\eta x \in p^{\rho}T^* \backslash p^{\rho}\eta T$. Now $0 \ne p^n x \in \langle a \rangle$ for some n so $h_p(p^k a) \ge \rho$ for some k , which is impossible. □

The information we now have leads to a number of necessary conditions for the admissibility of the pair (M, T^*) . Writing $\mathsf{M} = [\sigma_{pk}]$, it follows from (5.6) that

(a) *if* $\sigma_{pk} - 1$ *exists and a gap precedes* σ_{pk} *then*

$$P^{p}_{\sigma_{pk}-1}(T^*) \ne 0 .$$

Recall that our convention $\infty - 1 = \infty$ means that $\infty - 1$ exists. It is natural to ask how conditions *(i)-(iii)* of (5.7) can be translated to our situation. We see from (5.6) that condition *(i)* is taken care of automatically, while condition *(ii)* and (5.10) together imply

(b) *if* $\sigma_{pk} \ne \infty$ *then* $\sigma_{pk} < l_p(T^*) + \omega$.

5.11 LEMMA. *Let* A *have torsion free rank* 1 *and contain an element* a *of infinite order such that* $\mathsf{H}(a) = \mathsf{M}$ *and* $A/\langle a \rangle \cong T^*$. *Write* $T = T(A)$ *and* $\mathsf{M} = [\sigma_{pk}]$. *Then conditions (i) and (ii) of* (5.7) *are satisfied exactly when* (a) *and* (b) *are satisfied.*

Proof. We need only show *(ii)* is satisfied if and only if (b) is. In view of the fact that $l_p(T) + \omega = l_p(T^*) + \omega$ by (5.10), we have *(ii)* implies (b) and (b) implies the first part of *(ii)*. In the remaining case when (b) is assumed and $\sigma_{pk} = \infty$ for some but not all k , one considers the greatest integer l for which $\sigma_{pl} \neq \infty$ and recalls that (5.6) implies $f^{p}_{\sigma_{pl}}(T) \neq 0$ so $l_p(T) > \sigma_{pl}$. □

We will show that the awkward condition *(iii)* of (5.7) can be ignored completely when T^* is totally projective, and that (a) and (b) are alone sufficient for the admissibility of the pair (M, T^*) . We anticipate this with the following definition.

5.12 DEFINITION. Let $\mathsf{M} = [\sigma_{pk}]$ be a height matrix, T^* a torsion group. We say that M and T^* are *compatible* if they satisfy conditions (a) and (b) of (5.11). □

We now restrict attention to pairs (M, H^*) where H^* is totally projective. This enables us to use Ulm's theorem for totally projective groups in our calculations.

5.13 DEFINITION. Let A be the class of groups A such that A is an extension of a cyclic (finite or infinite) group by a totally projective group. An admissible pair (M, H^*) , where H^* is totally projective, is said to be A-*admissible*. If M is a height matrix and H^* is a torsion group then we say that the pair (M, H^*) is A-*compatible* if H^* is totally projective and M and H^* are compatible. □

Observe that A includes all torsion free groups of rank 1 , all totally projective groups (in fact a torsion group is a member of A exactly when it is totally projective), and by (2.26), groups with torsion free rank 1 having totally projective torsion part (the latter have been classified by Wallace [1]). However the torsion part of a group A in A need not be totally projective as the following well known example shows.

5.14 EXAMPLE (Nunke [3]). Given a prime p , Nunke constructs a group A such that $p^{\Omega}A \cong Z$ and

$$A/p^{\Omega}A \cong H_{\Omega}^{p}$$

(recall that H_σ^p is the generalised Prüfer p-group of length σ). It follows from (5.10) that the natural map $\eta : A \to A/p^\Omega A$ imbeds $T(A)$ as an Ω-dense isotype subgroup of H_Ω^p such that $H_\Omega^p/\eta T(A) \cong Z(p^\infty)$ - thus $T(A)$ is an Ω-elementary S-group. It is known that such a group cannot be totally projective (for example, see Warfield [2]). □

We list a number of elementary properties shared by groups in A .

(A). *If* $A \in \mathrm{A}$ *and* B *is a finitely generated subgroup of* A *then* $A/B \in \mathrm{A}$.

Since A is an extension

$$0 \to \langle a \rangle \to A \to A/\langle a \rangle \to 0 ,$$

the group A/B is the middle term of

$$0 \to \langle a+B \rangle \to A/B \to (A/B)/(\langle a, B \rangle/B) \to 0 .$$

Now

$$\begin{aligned}(A/B)/(\langle a, B\rangle/B) &\cong A/\langle a, B\rangle \\ &\cong (A/\langle a\rangle)/(\langle a, B\rangle/\langle a\rangle) \\ &= T^* \text{ , say.}\end{aligned}$$

Clearly $\langle a, B\rangle/\langle a\rangle$ is a finitely generated subgroup of the totally projective group $A/\langle a \rangle$; since (A) holds with A replaced by the class of totally projective groups (see Fuchs [2], p. 88), T^* is totally projective and $A/B \in \mathrm{A}$. In particular if g and h are elements of a group G such that $o(g) = o(h) = \infty$ and $G/\langle g \rangle$ is totally projective, then $G/\langle h\rangle$ is also totally projective.

(B) *If* $A \in \mathrm{A}$ *and the subgroup* C *of* A *has finite index in* A *then* $C \in \mathrm{A}$.

We use the fact that (B) holds for the class of totally projective groups in place of A (see Fuchs [2], p. 80); this also means that we need only consider the case where A has torsion free rank 1 . It follows that

C has torsion free rank 1 and, by (A), there is an element a in C such that $A/\langle a\rangle$ is totally projective. Then $C/\langle a\rangle$ has finite index in $A/\langle a\rangle$ and we are done.

Wallace [1] has shown that C is totally projective if A is totally projective and A/C is countable. We cannot similarly extend (B) to allow C to have countable index since in (5.14) the factor group $A/T(A)$ is countable although $T(A) \notin \mathsf{A}$. However the proof of (B) also shows:

(C) *If* $A \in \mathsf{A}$ *and the subgroup* C *of* A *is not torsion and has countable index in* A *then* $C \in \mathsf{A}$.

Although (5.9) shows that a general admissible pair $(\mathbb{M}, T^*)$ can be associated with non-isomorphic groups having torsion free rank 1 , such a possibility is excluded for A-admissible pairs by the following theorem.

5.15 THEOREM. *If* A *and* A' *are two groups in* A *containing, respectively, elements* a *and* a' *of infinite order such that* $\mathbb{H}(a) = \mathbb{H}(a')$ *and* $A/\langle a\rangle \cong A'/\langle a'\rangle$ *then* $A \cong A'$.

Proof. We may assume that A (and therefore A') is reduced. It is clear from (5.5) that

$$f_\sigma^{\,p}(A, \langle a\rangle) = f_\sigma^{\,p}(A', \langle a'\rangle)$$

for all primes p and ordinals σ , while $f_\infty^{\,p}(A, \langle a\rangle) = 0 = f_\infty^{\,p}(A', \langle a'\rangle)$ follows because A and A' are reduced. Now

$$f_\sigma^{\,p}\big(A(p, \langle a\rangle), \langle a\rangle\big) = f_\sigma^{\,p}(A, \langle a\rangle)$$

for all p and σ , so (2.14) yields, for each prime p , an isomorphism $A(p, \langle a\rangle) \to A'(p, \langle a'\rangle)$. The homomorphism $A \to A'$ given by (2.21) is clearly an isomorphism. □

With (5.15) in mind we make the following definition.

5.16 DEFINITION. Let $(\mathbb{M}, H^*)$ be an A-admissible pair and A the unique (up to isomorphism) group in $\mathcal{A}$ containing an element a of infinite order such that $\mathbb{H}(a) = \mathbb{M}$ and $A/\langle a \rangle \cong H^*$. Then we say that $(\mathbb{M}, H^*)$ *represents* A. □

One of the main reasons for considering groups with torsion free rank 1 as extensions is revealed by (5.15); we are able to classify groups in $\mathcal{A}$ using height matrices and the structure theory of totally projective groups even though their torsion parts need not be totally projective.

In general there are many different $\mathcal{A}$-admissible pairs representing the same group in $\mathcal{A}$, and some way of equating such pairs would be useful. We begin by defining invariants for an arbitrary pair $(\mathbb{M}, T^*)$ in a manner suggested by (5.6).

5.17 DEFINITION. Let $\mathbb{M} = [\sigma_{pk}]$ be a height matrix, T^* a torsion group. For each prime p, and for σ an ordinal or ∞ we define:

$$f_\sigma^p(\mathbb{M}, T^*) = \begin{cases} f_\sigma^p(T^*)+1 & \text{if } \sigma = \sigma_{pk} \text{ and a gap follows } \sigma_{pk}; \\ f_\sigma^p(T^*)-1 & \text{if there is an } n \text{ such that } \sigma+1 = \sigma_{pn} \text{ and a gap precedes } \sigma_{pn}; \text{ and} \\ f_\sigma^p(T^*) & \text{otherwise.} \end{cases}$$ □

5.18 DEFINITION. Let $(\mathbb{M}, H^*)$ and $(\mathbb{N}, G^*)$ be two $\mathcal{A}$-compatible pairs. We say that $(\mathbb{M}, H^*)$ and $(\mathbb{N}, G^*)$ are *equivalent* (in symbols - $(\mathbb{M}, H^*) \sim (\mathbb{N}, G^*)$) if $\mathbb{M} \sim \mathbb{N}$ and

$$f_\sigma^p(\mathbb{M}, H^*) = f_\sigma^p(\mathbb{N}, G^*)$$

for all ordinals σ and primes p. □

It is clear that $\sim$ defined in (5.18) is an equivalence relation - our next result shows that the obvious correspondence is a bijection between equivalence classes of $\mathcal{A}$-admissible pairs and the isomorphism classes in $\mathcal{A}$.

5.19 THEOREM. *Let* A, A' *be two groups in* $\mathcal{A}$ *containing, respectively, elements* a, a' *of infinite order such that* $\mathsf{H}(a) = \mathsf{M}$ *and* $\mathsf{H}(a') = \mathsf{N}$. *Then* $A \cong A'$ *if and only if* $(\mathsf{M}, A/\langle a\rangle) \sim (\mathsf{N}, A'/\langle a'\rangle)$.

Proof. Suppose $(\mathsf{M}, A/\langle a\rangle) \sim (\mathsf{N}, A'/\langle a'\rangle)$. It is clear from (5.6) that the Ulm invariants of A and A' are the same. Replacing a and a', if necessary, by suitable multiples of themselves we may assume $\mathsf{H}(a) = \mathsf{H}(a')$. Then (5.6) and (2.14) yield $A/\langle a\rangle \cong A'/\langle a'\rangle$. By (5.15) $A \cong A'$. The converse is obvious. □

We require the following two lemmas for $\mathcal{A}$-compatible pairs.

5.20 LEMMA. *Let* (M, H^*) *and* (N, G^*) *be* $\mathcal{A}$*-compatible pairs. If* $(\mathsf{M}, H^*) \sim (\mathsf{N}, G^*)$ *and* (M, H^*) *is* $\mathcal{A}$*-admissible then so is* (N, G^*).

Proof. Let A be a group in $\mathcal{A}$ represented by (M, H^*) and let a in A be such that $\mathsf{H}(a) = \mathsf{M}$. If we can find an a' in A such that $\mathsf{H}(a') = \mathsf{N}$ then (5.6) shows that the Ulm invariants of $T^* = A/\langle a'\rangle$ and G^* are the same and (2.14) yields $T^* \cong G^*$. Thus (N, G^*) is $\mathcal{A}$-admissible.

We reduce the problem of finding such an a' to the case $p\mathsf{N} = \mathsf{M}$ for some prime p as follows. Since $\mathsf{M} \sim \mathsf{N}$, there are integers k, $l \geq 1$ such that $k\mathsf{M} = l\mathsf{N}$ and since $(k\mathsf{M}, A/\langle ka\rangle)$ represents A and is equivalent to (M, H^*) then $(k\mathsf{M}, A/\langle ka\rangle) \sim (\mathsf{N}, G^*)$. Thus we can assume $l\mathsf{N} = \mathsf{M}$ for some integer $l > 0$; there is an immediate reduction to the case $p\mathsf{N} = \mathsf{M}$ where p is prime.

Let us now find a'. Write $\mathsf{N}_p = (\sigma_0, \sigma_1, \ldots)$ so that $h_p(a) = \sigma_1$; we require a' in A such that $h_p(a') = \sigma_0$ and $pa' = a$. When $\sigma_0 + 1 = \sigma_1$ there is no problem, and when $\sigma_0 + 1 < \sigma_1$, it is clear that such an a' will exist exactly when $f^p_{\sigma_0}(A) \neq 0$. To see that this condition is satisfied, suppose $\sigma_0 + 1 < \sigma_1$. Then

$$f_{\sigma_0}^{\,p}(A) = f_{\sigma_0}^{\,p}(\mathsf{M},\ H^*)$$

$$= f_{\sigma_0}^{\,p}(\mathsf{N},\ G^*)$$

$$= f_{\sigma_0}^{\,p}(G^*) + 1$$

$$\neq 0\ . \qquad \square$$

5.21 LEMMA. *Let* $(\mathsf{M},\ H^*)$ *be an* A*-compatible pair. Then for each integer* $n \geq 1$ *there is a totally projective group* G^* *such that* $(n\mathsf{M},\ G^*)$ *is* A*-compatible and* $(n\mathsf{M},\ G^*) \sim (\mathsf{M},\ H^*)$.

Proof. There is no loss of generality in assuming that $n = p$ and that H^* is a p-group for some prime p . Suppose $\mathsf{M}_p = (\sigma_0,\ \sigma_1,\ \ldots)$ and define a function g as follows: if σ is an ordinal then

$$g(\sigma) = \begin{cases} f_\sigma^{\,p}(H^*)-1 & \text{if } \sigma+1 = \sigma_0\ ; \\ f_\sigma^{\,p}(H^*)+1 & \text{if } \sigma = \sigma_0\ ; \\ f_\sigma^{\,p}(H^*) & \text{otherwise;} \end{cases}$$

and if $\sigma = \infty$ then $g(\sigma) = f_\sigma^{\,p}(H^*)$.

Now $(\mathsf{M},\ H^*)$ is A-admissible so that condition (a) of (5.12) ensures $f_\sigma^{\,p}(H^*) \neq 0$ if $\sigma + 1 = \sigma_0$, and therefore $g(\sigma) \geq 0$ for all σ . Since we have only made finite changes in finitely many places to the Ulm invariants of H^* to obtain g , it is clear that g is admissible. By (2.16) there is a totally projective p-group G^* with $g(\sigma) = f_\sigma^{\,p}(G^*)$ for all σ .

For compatibility of $(p\mathsf{M},\ G^*)$ observe that $l_p(G^*) \geq l_p(H^*)$ so that $p\mathsf{M}$ and G^* satisfy (b) with $p\mathsf{M}$ in place of M and G^* in place of T^* , while (a) needs checking only when $\sigma_0 + 1 = \sigma_1$. We then require

$f_{\sigma_0}^{p}(G^*) \neq 0$ which is guaranteed by our choice of g. As $p\mathsf{M} \sim \mathsf{M}$ it remains only to show $f_\sigma^{\,p}(\mathsf{M}, H^*) = f_\sigma^{\,p}(p\mathsf{M}, G^*)$ for all σ. This is obvious except when $\sigma \neq \infty$ and $\sigma + 1 = \sigma_0$ or $\sigma = \sigma_0$. We distinguish three cases and run through the calculation explicitly.

1. If $\sigma + 1 = \sigma_0$ then

$$\begin{aligned} f_\sigma^{\,p}(p\mathsf{M}, G^*) &= f_\sigma^{\,p}(G^*) \\ &= f_\sigma^{\,p}(H^*) - 1 \\ &= f_\sigma^{\,p}(\mathsf{M}, H^*)\ . \end{aligned}$$

2. If $\sigma = \sigma_0$ and $\sigma_0 + 1 = \sigma_1$ then

$$\begin{aligned} f_\sigma^{\,p}(p\mathsf{M}, G^*) &= f_\sigma^{\,p}(G^*) - 1 \\ &= f_\sigma^{\,p}(H^*) \\ &= f_\sigma^{\,p}(\mathsf{M}, H^*)\ . \end{aligned}$$

3. If $\sigma = \sigma_0$ and $\sigma_0 + 1 < \sigma_1$ then

$$\begin{aligned} f_\sigma^{\,p}(p\mathsf{M}, G^*) &= f_\sigma^{\,p}(G^*) \\ &= f_\sigma^{\,p}(H^*) + 1 \\ &= f_\sigma^{\,p}(\mathsf{M}, H^*)\ . \qquad \square \end{aligned}$$

We show that A-compatible pairs are A-admissible in two stages - first for a restricted class of A-compatible pairs and using this, for an arbitrary A-compatible pair.

5.22 THEOREM. *Suppose* M *is a height matrix with only a finite number of gaps. Then a pair* (M, H^*) *is* A-*admissible if and only if it is* A-*compatible.*

Proof. From the discussion leading up to (5.12), it is immediate that all A-admissible pairs are A-compatible. For the converse part, suppose (M, H^*) is A-compatible. There is an integer $n \geq 1$ such that $n\mathsf{M}$ has no gaps; now (5.21) lets us replace M by $n\mathsf{M}$ and H^* by G^* such that $(n\mathsf{M}, G^*) \sim (\mathsf{M}, H^*)$ while (5.20) shows (M, H^*) is A-admissible if and only if $(n\mathsf{M}, G^*)$ is. In view of this we assume M contains no gaps and by (5.2) we replace M by $\mathsf{M}_p = \mathsf{u} = (\sigma, \sigma+1, \sigma+2, \ldots)$ and H^* by H_p^*. Next we show that $l_p(H^*) = \sigma$ can also be assumed without loss of generality. Since A-compatibility of (u, H^*) ensures that $f^p_{\sigma-1}(H^*) \neq 0$ when σ is not a limit ordinal, (2.18) allows us to write $H^* = H_1^* \oplus H_2^*$ with $l_p(H_1^*) = \sigma$. Now (u, H_1^*) is A-compatible and if (u, H_1^*) represents A' then (u, H^*) represents $A = A' \oplus H_2^*$. As the result is trivial when $\sigma = \infty$, we also assume that σ is an ordinal.

We use a construction of Hill and Megibben [1]: it follows from (2.2) that there is a subgroup H of H^* such that H is isotype and σ-dense in H^* and $0 \neq H^*/H = Z(p^\alpha)$ where $\alpha \in \{0, 1, \ldots\} \cup \{\infty\}$. As we observed in the proof of (2.2), $\alpha = \infty$ whenever $\sigma \geq \omega$. There is a torsion free group R of rank 1 containing an element a such that $h_p^R(a) = \alpha$ and such that $H^*/H \cong R/\langle a \rangle$. We define A to be the subdirect sum of H^* and R having kernels H and $\langle a \rangle$ respectively. That is, if $\eta : H^* \to H^*/H$ and $\delta : R \to R/\langle a \rangle$ are the natural homomorphisms then $A = \{h+r \in H^* \oplus R : \eta h = \delta r\}$. Identifying H^* and R as subgroups of $H^* \oplus R$ in the natural way, we have $A + H^* = A + R = H^* \oplus R$ and $A \cap H^* = H$ and $A \cap R = \langle a \rangle$. Arguing as in Proposition 1.7 of Hill and Megibben [1] we have

$$p^\sigma A = A \cap p^\sigma(H^* \oplus R) .$$

Now $p^\sigma(H^* \oplus R) = p^\sigma R$ and since $p^\sigma R = \langle a \rangle$ when $\sigma < \omega$ and $p^\sigma R = R$ when

$\sigma \geq \omega$, it follows that $p^\sigma A = A \cap p^\sigma R = \langle a \rangle$. Therefore $h_p^A(a) = \sigma$ and

$$A/\langle a \rangle = A/A \cap R = (A+R)/R = (H^* \oplus R)/R \cong H^*$$

shows that A is represented by (u, H^*) . □

It will be convenient to have a name for the groups described in (5.22).

5.23 DEFINITION. Let C be the class of groups C in A such that the height matrix of every element c in C contains only finitely many gaps. □

Before proceeding further, we point out a modification to (5.7). Megibben [2] has shown that if λ is a limit ordinal not cofinal with ω and if $T = \oplus G_\sigma$ has p-length λ and each G_σ is a p-group with p-length less than λ , then T is the torsion part of the inverse limit

$$\varprojlim T/p^\rho T \quad (\rho \to \lambda) .$$

Now a totally projective p-group T with p-length a limit ordinal λ can always be written as a direct sum of totally projective groups with p-lengths less than λ (see Fuchs [2], p. 70). Further, for each ordinal $\gamma < \lambda$ the groups $p^\gamma T$ and $T/p^\gamma T$ are always totally projective (and therefore never torsion complete) so that for a group A with torsion free rank 1 and totally projective torsion part T , condition *(iii)* of (5.7) becomes

(iv) *if* λ *is a limit ordinal not cofinal with* ω *and if* $\lambda \leq l_p(T) < \lambda + \omega$ *then* $\sigma_{pk} \neq \infty$ *implies*

$$\sigma_{pk} < \max\left(l_p(T_p), \omega\right) .$$

Thus (5.7) gives:

5.24 THEOREM. *Let* T *be a reduced totally projective group,* M *a height matrix. There exists a mixed group* A *of torsion free rank* 1 *with* $T(A) = T$ *and which contains an element* a *of infinite order with* $\mathsf{H}(a) = \mathsf{M}$

if and only if M *satisfies (i) and (ii) of* (5.7) *and (iv).* □

Now the main theorem of this chapter:

5.25 THEOREM. *A reduced pair* (M, H^*) *is* A*-admissible if and only if it is* A*-compatible.*

Proof. Only half the statement requires verification, and again we can replace M by $\mathsf{M}_p = \mathrm{u} = (\sigma_0, \sigma_1, \ldots)$ and H^* by $H_p{}^*$. Assume that (u, H^*) is A-compatible. In view of (5.22) we need only consider the case when u has infinitely many gaps. It is easy to see, using (2.16), that there is a totally projective p-group H with Ulm invariants given by

$$f_\sigma{}^p(H) = f_\sigma{}^p(\mathsf{M}, H^*) .$$

Evidently $l_p(H) > \sigma_n$ for $n = 0, 1, \ldots$ so that u and H satisfy *(iv)* of (5.24) with u in place of M and H in place of T. Let A be the group given by (5.24) with torsion part H and element a of infinite order such that $U_p(a) = \mathrm{u}$. Since $A/\langle a \rangle$ is totally projective and has the same Ulm invariants as H^*, it follows that $A/\langle a \rangle \cong H^*$. □

Having solved the problem of which pairs (M, H^*) are admissible when H^* is totally projective, it would be useful to know whether every height matrix occurs in some A-admissible pair.

5.26 PROPOSITION. *To each height* $\mathrm{K} = \langle\langle \beta_2, \beta_3, \ldots, \beta_p, \ldots \rangle\rangle$ *there is a group* C *in* C *containing an element* c *of infinite order such that* $\mathrm{H}(c) = \mathrm{K}$. *Further, this* c *can be chosen so that* $\mathsf{H}(c)$ *contains no gaps.*

Proof. Let M be the height matrix with no gaps and first column $\langle\langle \beta_2, \ldots, \beta_p, \ldots \rangle\rangle$ and let $H^* = \bigoplus_{p \in P} H_{\beta_p}^p$, where $H_{\beta_p}^p$ is the generalised Prüfer p-group of length β_p. It is easy to check that (M, H^*) is

A-compatible and so (5.22) yields the result. □

5.27 PROPOSITION. *To each height matrix* M *there is a group* A *in* $\mathcal{A}$ *represented by* (M, H^*) *for some totally projective group* H^* .

Proof. Let $\mathsf{M} = [\sigma_{pk}]$ and set

$$H^* = \bigoplus_{p\in P} \bigoplus_{k<\omega} H_{\sigma_{pk}}^{p}$$

and argue as for (5.26). □

Let us now turn to a study of the torsion part of a group in $\mathcal{A}$.

5.28 THEOREM. *If* $A \in \mathcal{A}$ *then* $T(A)$ *is an* S*-group.*

Proof. We need only consider a reduced group A in $\mathcal{A}$ with torsion free rank 1 represented by (M, H^*) where H^* is a p-group. Put $u = \mathsf{M}_p = (\sigma_0, \sigma_1, \ldots)$, let $\eta : A \to H^*$ be the natural homomorphism and T be the torsion part of A . Recall that $\eta T \cong T$. We consider three cases.

1. u has infinitely many gaps. It is easy to see, using (2.16), that there is a totally projective p-group H with Ulm invariants given by

$$f_\sigma^{\,p}(H) = f_\sigma^{\,p}(\mathsf{M}, H^*) .$$

Arguing as in the proof of (5.25) we see that M and H satisfy condition *(iv)* of (5.24) with T replaced by H . Since conditions *(i)* and *(ii)* of (5.7) are automatically satisfied, (5.24) gives a group A' with torsion free rank 1 such that $T(A') = H$ and containing an element a' with $\mathsf{H}(a') = \mathsf{M}$. Now (5.6) implies $A'/\langle a' \rangle \cong H^*$ and thus $A' \in \mathcal{A}$. By (5.15), $A \cong A'$ and so $T \cong T(A') = H$.

2. u has only finitely many gaps and $\sup_i \sigma_i = \rho < \infty$. Let λ be the limit ordinal such that $\lambda + \omega = \rho$. When λ is cofinal with ω it follows, using the argument of 1 , that T is totally projective. Suppose that λ is not cofinal with ω . By changing to another representation if

necessary, we can assume $\sigma_0 \geq \lambda$. Note that the new H^* obtained from any such change is still totally projective, while T is of course unaffected. Now (5.10) shows that ηT is λ-dense in H^* , and that

$$p^{\sigma}\eta T = p^{\sigma}H^* \cap \eta T \quad \text{for all} \quad \sigma \leq \lambda \ .$$

Hence

$$\eta T/p^{\lambda}\eta T = \eta T/\left(p^{\lambda}H^*\cap\eta T\right) \cong \left(\eta T+p^{\lambda}H^*\right)/p^{\lambda}H^*$$

and we now prove that $\eta T/p^{\lambda}\eta T$ is an S-group by showing that $\left(\eta T+p^{\lambda}H^*\right)/p^{\lambda}H^*$ is λ-dense and isotype in the totally projective p-group $H^*/p^{\lambda}H^*$. For each ordinal $\sigma < \lambda$ we have

$$\begin{aligned}\left(\eta T+p^{\lambda}H^*\right)/p^{\lambda}H^* + p^{\sigma}\left(H^*/p^{\lambda}H^*\right) &= \left(\eta T+p^{\sigma}H^*\right)/p^{\lambda}H^* \\ &= H^*/p^{\lambda}H^*\end{aligned}$$

as ηT is λ-dense in H^* . On the other hand, for each ordinal $\sigma \leq \lambda$ we have

$$\begin{aligned}p^{\sigma}\left(\left(\eta T+p^{\lambda}H^*\right)/p^{\lambda}H^*\right) &\leq \left(\eta T+p^{\lambda}H^*\right)/p^{\lambda}H^* \cap p^{\sigma}\left(H^*/p^{\lambda}H^*\right) \\ &= \left(\left(\eta T+p^{\lambda}H^*\right) \cap p^{\sigma}H^*\right)/p^{\lambda}H^* \\ &= \left(p^{\lambda}H^*+\left(\eta T\cap p^{\sigma}H^*\right)\right)/p^{\lambda}H^* \quad \text{by the modular law} \\ &= \left(p^{\lambda}H^*+p^{\sigma}\eta T\right)/p^{\lambda}H^* \qquad \text{by (5.10)} \\ &\leq p^{\sigma}\left(\left(\eta T+p^{\lambda}H^*\right)/p^{\lambda}H^*\right)\end{aligned}$$

so that

$$p^{\sigma}\left(\left(\eta T+p^{\lambda}H^*\right)/p^{\lambda}H^*\right) = \left(\eta T+p^{\lambda}H^*\right)/p^{\lambda}H^* \cap p^{\sigma}H^*/p^{\lambda}H^* \ .$$

Since the p-length of $H^*/p^{\lambda}H^*$ is λ , this shows that $\left(\eta T+p^{\lambda}H^*\right)/p^{\lambda}H^*$ is indeed isotype in $H^*/p^{\lambda}H^*$. It follows from (5.10) that $p^{\rho}\eta T = p^{\rho}H^*$ so that $p^{\lambda}\eta T/p^{\rho}\eta T$ is a subgroup of $p^{\lambda}H^*/p^{\rho}H^*$. As $\lambda + \omega = \rho$ we have that $p^{\lambda}H^*/p^{\rho}H^*$ is a totally projective p-group whose p-length is at most ω . Therefore $p^{\lambda}H^*/p^{\rho}H^*$ is a direct sum of cyclic p-groups. This means that $p^{\lambda}\eta T/p^{\rho}\eta T$ is also a direct sum of p-groups. Put $G = \eta T/p^{\rho}\eta T$; then we have shown that $p^{\lambda}G$ and $G/p^{\lambda}G$ are both S-groups. Therefore G is an S-group. However $p^{\rho}\eta T = p^{\rho}H^*$ is also an S-group so that ηT is itself an S-group.

3. u contains ∞ . Assume $\sigma_0 = \infty$; then H^* has a unique summand $Z(p^{\infty})$ such that $\eta T \cap Z(p^{\infty}) = 0$. In the proof of (5.10) we saw that $\eta T + Z(p^{\infty}) = H^*$ so that $H^* = \eta T \oplus Z(p^{\infty})$. Thus ηT is totally projective. □

In particular the only case when the p-component of an A in A is not automatically totally projective is when there is an element a in A of infinite order such that $U_p(a)$ does not contain ∞ and has only finitely many gaps. We conclude our investigation of the torsion part of a group A in A by giving two alternative conditions under which A is a direct sum of a p-group and a group whose p-component is 0 or has limit ordinal p-length.

5.29 PROPOSITION. *Let A be a reduced group in A represented by (M, H^*) , and $\mathsf{M}_p = \mathsf{u} = (\sigma_0, \sigma_1, \ldots)$. If u has only finitely many gaps, or if $\sigma_{i+1} \geq \sigma_i + \omega$ for infinitely many values of i , then*

$$A = A' \oplus H$$

where H is a totally projective p-group and $l_p(A_p')$ is a limit ordinal or 0 .

Proof. We can assume H^* (and therefore $T(A)$) is a p-group. We deal with the two cases separately.

1. u has only finitely many gaps. Let the last gap precede σ_n . In the case where $\sigma_n \neq \infty$, let λ be the limit ordinal (or 0) such that $\lambda + k = \sigma_n$ for some non-negative integer k . An easy modification of the

proof of (2.18) yields a summand G^* of H^* compatible with $\mathfrak{M}$ and such that when $\sigma \geq \lambda$,

$$f_\sigma^{\,p}(G^*) = \begin{cases} 1 & \text{if } \sigma+1 = \sigma_i \text{ and a gap precedes } \sigma_i \text{; and} \\ 0 & \text{otherwise.} \end{cases} \tag{14}$$

In the case $\sigma_n = \infty$ and $n = 0$ we put $G^* = Z(p^\infty)$, otherwise $\sigma_{n-1} \neq \infty$ and there is a limit ordinal λ and integer $k \geq 0$ such that $\lambda + k = \sigma_{n-1}$. Again we can choose a summand G^* of H^* compatible with $\mathfrak{M}$ and satisfying (14). Write $H^* = G^* \oplus H$ and suppose A' is represented by $(\mathfrak{M}, G^*)$. Then $A' \oplus H$ is represented by $(\mathfrak{M}, H^*)$ and we have $A' \oplus H \cong A$. Clearly $l_p(A_p') = \lambda$.

2. $\mathfrak{u}$ has infinitely many infinite gaps. Write $\rho = \sup_i \sigma_i$. Since there are only finitely many σ_i in any ω-type interval, it is again possible to modify (2.18) to obtain a summand G^* of H^* such that $l_p(G^*) = \rho$ and such that $f_\sigma^{\,p}(G^*) \neq 0$ whenever $\sigma + 1 = \sigma_i$ and a gap precedes σ_i. Clearly $(\mathfrak{M}, G^*)$ is compatible and if $(\mathfrak{M}, G^*)$ represents A' then arguing as above we have $A = A' \oplus H$ for some torsion H, while $l_p(A_p') = \rho$. □

The following example demonstrates that (5.29) cannot be strengthened to include all groups in $\mathbb{A}$.

5.30 EXAMPLE. Let p be a prime and $\mathfrak{M}$ a height matrix such that $\mathfrak{M}_p = (0, 2, 4, \ldots)$ and $\mathfrak{M}_q = (0, 1, 2, \ldots)$ for all primes $q \neq p$. Let T be the totally projective p-group whose non-zero Ulm invariants are $f_{2n}^{\,p}(T) = 1$ for $n = 0, 1, \ldots$ and $f_\omega^{\,p}(T) = 1$. By (5.24) there is a group A in $\mathbb{A}$ with $T(A) = T$ and an element a in A of infinite order such that $\mathbb{H}(a) = \mathfrak{M}$. If $A = A' \oplus H$ and $l_p(A_p') = \omega$ (this is the only limit ordinal possibility for $l_p(A_p')$) then $l_p(H) = \omega + 1$ while (5.6) implies that $f_{\sigma}^{p}(H) = 0$ for almost all σ such that $0 \leq \sigma < \omega$, which is impossible. $\square$

We conclude the chapter with the proof of a result promised in Chapter 2.

5.31 THEOREM. *Let* $\mathbb{M}$ *be a height matrix. Then* $S_{\mathbb{M}}$ *is a cotorsion functor. Moreover,* $S_{\mathbb{M}}$ *has enough projectives exactly when* $\mathbb{M}$ *contains no gaps.*

Proof. That $S_{\mathbb{M}}$ is a cotorsion functor follows from (2.25) and (5.27). If $\mathbb{M}$ contains no gaps and $\langle\langle \beta_2, \beta_3, \ldots, \beta_p, \ldots\rangle\rangle$ is the first column of $\mathbb{M}$ then the proof of (5.26) yields a representing sequence

$$0 \to \langle a \rangle \to A \to H^* \to 0$$

for $S_{\mathbb{M}}$ such that H^* is totally projective and $p^{\beta_p} H_p{}^* = 0$ for all p in P. Since $p^{\beta_p}\operatorname{Ext}(H_p{}^*, X) = 0$ for all p in P and all groups X, it follows that

$$\operatorname{Ext}(H^*, X)(\mathbb{M}) = \bigcap_{p \in P} p^{\beta_p}\operatorname{Ext}(H^*, X) = 0 .$$

Thus H^* is $S_{\mathbb{M}}$-projective and $S_{\mathbb{M}}$ has enough projectives.

On the other hand, suppose $\mathbb{M}_p = (\sigma_0, \sigma_1, \ldots, \sigma_n, \ldots)$ contains a gap which follows σ_n, say. Choose a group A containing an element a such that $U_p(a) = (\sigma_0, \sigma_1, \ldots, \sigma_n, \sigma_n+1, \infty, \ldots)$ and $U_q(a) = \mathbb{M}_q$ when $q \neq p$. Clearly

$$0 \neq a + A(\mathbb{M}) \in [A/A(\mathbb{M})](\mathbb{M})$$

so that $S_{\mathbb{M}}$ is not a radical. $\square$

CHAPTER 6

BALANCED PROJECTIVES

In this chapter we characterise the balanced projectives in the category of all Abelian groups: they are just the direct summands of direct sums of members of the class A. We also explore the properties of balanced projectives. Every group is the balanced image of a balanced projective. If A is balanced projective, then $A/T(A)$ is completely decomposable and $T(A)$ is a summand of an S-group. A torsion (torsion free) group is balanced projective in the category of all Abelian groups if and only if it is totally projective (completely decomposable).

The class of balanced sequences of torsion groups is closed under the functors S_{M}, and the corresponding class of (torsion) projectives (the totally projective groups) is similarly closed under the S_{M} (see 2.19). Moreover, the class of balanced sequences of arbitrary groups is, by definition, the largest class of short exact sequences closed under the functors S_{M}. It is therefore somewhat surprising that (as an example will show) the class of balanced projectives is not also closed under the S_{M}. However, we show that the closure of the class of balanced projectives under the S_{M} (as well as direct sums and direct summands) is in fact the class of projectives relative to a suitably chosen class of balanced sequences which we call C-balanced. The essential difference between a balanced sequence and a C-balanced sequence is shown to occur with the torsion; we use this to show that the C-balanced sequences are also closed under the functors S_{M}. Our main result on C-balanced projectives (we use the term C-projective) is (6.20), namely: every C-projective is a direct summand of a direct sum of groups of the form $A(\mathrm{M})$ where A is balanced projective and M is a height matrix, and if A is balanced projective and M is a height matrix then $A(\mathbb{M})$ is C-projective. We also show that if A is balanced projective and $\mathbb{M}$ is a height matrix then $A/A(\mathbb{M})$ is balanced projective. These results enable us to show that, given an S-group A and height matrix $\mathbb{M}$, both $A(\mathbb{M})$ and $A/A(\mathbb{M})$ are S-groups.

Although we do not provide the details here, the arguments in Chapter 4 readily adapt to show that the C-balanced injectives are just the pure injectives. Similarly, most of the results of Chapter 3 carry over with C-balanced replacing balanced.

We begin with a generalisation of Lemma 80.3 of Fuchs [2] to mixed groups.

6.1 LEMMA. *Given a commutative diagram*

$$\begin{array}{ccccccccc} 0 & \to & N & \longrightarrow & A & & \\ & & \downarrow{\scriptstyle\psi} & & \downarrow{\scriptstyle\phi} & & \\ 0 \to U \to & & V & \xrightarrow{\alpha} & W & \to & 0 \end{array}$$

where U *is* H*-balanced in* V *and the two rows are exact, suppose that* ψ *does not decrease heights in* A. *If the element* a *is* p*-proper with respect to* N *and* $pa \in N$ *then* ψ *can be extended to a map*

$$\psi^* : \langle N, a\rangle \to V$$

such that $\alpha\psi^* a = \phi a$ *and* ψ^* *does not decrease heights.*

Proof. If $\mathrm{H}(a) = \langle\langle \beta_2, \beta_3, \ldots, \beta_p, \ldots\rangle\rangle = \mathrm{K}$ then some v in $V(\mathrm{K})$ satisfies $\alpha v = \phi a$. Write $\mathrm{K}' = \langle\langle \beta_2, \ldots, \beta_p+1, \ldots\rangle\rangle$ - that is, K' is the same as K except in the p-th entry, which is increased by 1. Clearly $pv - \psi pa \in U \cap V(\mathrm{K}') = U(\mathrm{K}')$ so there is a u in $U(\mathrm{K})$ with $pu = pv - \psi pa$. We extend ψ to a homomorphism ψ^* of $\langle N, a\rangle$ into V by setting $\psi^* a = v - u \in V$: this clearly satisfies $\alpha\psi^* a = \alpha v = \phi a$. Now $\mathrm{H}(v-u) \geq \mathrm{K} = \mathrm{H}(a)$ and to check that ψ^* does not decrease heights it is enough to show that $\mathrm{H}(ma+x) \leq \mathrm{H}\big(m(v-u)+\psi x\big)$ whenever $x \in N$ and $1 \leq m < p$. Since a, and therefore ma, is p-proper with respect to N we have $h_p(ma+x) = \min\big(h_p(ma), h_p(x)\big)$, while

$$h_p(mv-mu+\psi x) \geq \min\left(h_p\left(m(v-u)\right), h_p(\psi x)\right)$$
$$\geq \min\left(h_p(ma),\ h_p(x)\right) .$$

If $q \neq p$ then

$$h_q(ma+x) = h_q(pma+px)$$
$$\leq h_q\left(pm(v-u)+\psi px\right) \quad \text{as } \psi \text{ does not decrease heights}$$
$$= h_q\left(m(v-u)+\psi x\right) . \quad \square$$

6.2 THEOREM. *If* $A \in \overline{\mathrm{A}}$ *then* A *is balanced projective, and if* $C \in \overline{\mathrm{C}}$ *then* C *is* H*-projective.*

Proof. Since direct sums and direct summands of projectives are again projective, we need only consider groups A in A and C in C.

Suppose $A \in \mathrm{A}$, let $\phi : A \to W$ be a homomorphism and let

$$0 \to U \to V \xrightarrow{\alpha} W \to 0$$

be a balanced sequence; our task is to show that ϕ lifts to a homomorphism $\psi : A \to V$ such that $\alpha\psi = \phi$. When A is torsion, A is totally projective. Now $\phi A \leq T(W)$ and since

$$0 \to T(U) \to T(V) \to T(W) \to 0$$

is balanced (see (3.10)), ϕ lifts to $\psi : A \to T(V)$ such that $\alpha\psi = \phi$. As ψ can be considered as a homomorphism $\psi : A \to V$, we have the required lifting. If A has torsion free rank 1, let a be an element of infinite order in A and choose v in V such that $\alpha v = \phi a$ and $\mathrm{H}(\phi a) = \mathrm{H}(v)$. The correspondence $a \longmapsto v$ gives rise to a homomorphism $\psi' : \langle a \rangle \to V$ which does not decrease heights in A. If ϕ_p is the restriction of ϕ to $A(p, \langle a \rangle)$ for each prime p, we have the commutative diagram

$$\begin{array}{ccccccccc}
0 \to & \langle a \rangle & \longrightarrow & A(p, \langle a \rangle) & \to & (A/\langle a \rangle)_p & \to 0 \\
& \downarrow{\scriptstyle \psi'} & & \downarrow{\scriptstyle \phi_p} & & & \\
0 \to U \to & V & \xrightarrow{\alpha} & W & \to & 0 &
\end{array}$$

where both rows are exact and the bottom row is balanced. Now (2.24) implies

$\langle a \rangle$ is p-nice in $A(p, \langle a \rangle)$; using (6.1) and a nice composition series for $(A/\langle a \rangle)_p$ we extend ψ' to $\psi_p : A(p, \langle a \rangle) \to V$ in such a way that $\alpha\psi_p = \phi_p$ (this extension is done transfinitely, taking unions at limit ordinals and using (6.1) at non-limit ordinals). An application of (2.20) yields a $\psi : A \to V$ such that $\alpha\psi = \phi$.

The proof that groups C in C , and hence groups in $\overline{\mathsf{C}}$, are H-projective is similar: the case for torsion C is the same as above, while for C having torsion free rank 1 we must choose a c in C with infinite order such that $\mathsf{H}(c)$ has no gaps to ensure that the map $\psi' : \langle c \rangle \to V$ does not decrease heights in C . □

6.3 THEOREM. *There are enough balanced* (H-*balanced) projectives. In particular, every group* G *can be imbedded in a balanced* (H-*balanced) sequence* $0 \to B \to A \to G \to 0$ *where* $A \in \mathsf{A}^{\Sigma}$ $(A \in \mathsf{C}^{\Sigma})$ *, and every balanced* (H-*balanced) projective is in* $\overline{\mathsf{A}}$ $(\overline{\mathsf{C}})$.

Proof. Corresponding to each element g in G we choose a group A_g in A containing an element a_g such that $o(a_g) = o(g)$ and $\mathsf{H}(a_g) = \mathsf{H}(g)$; when $o(g) = \infty$ we refer to (5.27) and for the case when $o(g) \neq \infty$ one constructs, as in the proof of (3.9), a finite direct sum of generalised Prüfer groups containing the required element. By (2.15), the height preserving map $\phi : \langle a_g \rangle \to \langle g \rangle$ sending $a_g \mapsto g$ extends to a map $A_g(p, \langle a_g \rangle) \to G$ and (2.21) provides a homomorphism $\phi_g : A_g \to G$ whose restriction to $\langle a_g \rangle$ is ϕ . The epimorphism

$$\bigoplus_{g \in G} \phi_g : \bigoplus_{g \in G} A_g \to G$$

satisfies condition (b) of (3.12) and is therefore balanced. An identical argument can be used to show the existence of enough H-projectives.

Finally, if G is balanced projective then the above argument provides a group A in A^{Σ} and a balanced epimorphism $\phi : A \to G$. Since G is balanced projective, there is $\psi : G \to A$ such that the diagram

$$\begin{tikzcd} & G \arrow[dl, "\psi"'] \arrow[d, equal] \\ A \arrow[r, "\phi"'] & G \end{tikzcd}$$

commutes. Hence G is a summand of A. □

Combining (6.2) and (6.3) we get:

6.4 THEOREM. *(i)* A *is balanced projective if and only if* $A \in \overline{\mathrm{A}}$.

(ii) A *is* H*-projective if and only if* $A \in \overline{\mathrm{C}}$. □

Another interesting property of balanced projectives is given in:

6.5 PROPOSITION. *An exact sequence*

$$0 \to B \to A \xrightarrow{\alpha} C \to 0$$

of groups relative to which every group in A (C) *has the projective property is balanced* (H*-balanced*).

Proof. Let c be an arbitrary element of C. The proof of (6.3) gives a reduced group G in A containing an element g such that $\mathrm{H}(g) = \mathrm{H}(c)$, $o(g) = o(c)$ and a homomorphism $\phi : G \to C$ for which $\phi g = c$. By assumption there is a homomorphism $\psi : G \to A$ such that $\alpha\psi = \phi$. Writing $a = \psi g$ we have $\alpha a = c$, $o(a) = o(c)$ and $\mathrm{H}(a) = \mathrm{H}(c)$. Thus B is balanced in A. The proof for H-balanced is identical. □

A summand of a balanced projective is again balanced projective. If A is balanced projective then $A/T(A)$ is completely decomposable; this follows from the fact that A is a direct summand of a direct sum of groups having torsion free rank 1. We see from (6.4) and (5.28) that $T(A)$ is a summand of an S-group. These results are summarised in the following:

6.6 PROPOSITION. *(i) The torsion part of every balanced projective*

(H-projective) is in $\overline{S}$.

(ii) A torsion (torsion free) group is balanced projective if and only if it is totally projective (completely decomposable); the same holds for H-balanced.

(iii) If A *is balanced projective (H-projective) then* $A/T(A)$ *is completely decomposable.*

(iv) A torsion summand of a balanced (H-balanced) projective is totally projective. □

We now give the promised example which shows that the class of balanced projectives is not closed under the functors S_{M} .

6.7 EXAMPLE. Let A be the group mentioned in (5.14) which satisfies $p^{\Omega}A \cong Z$ and $A/p^{\Omega}A \cong H_{\Omega}^{p}$ for a given prime p . We saw that the torsion part of A is an Ω-elementary S-group and is therefore not totally projective. For each prime $q \neq p$, the q-height of every element of infinite order in A is finite. Wick [1] has observed that, for primes $q \neq p$,

$$q^{\infty}A = q^{\omega}A = S_{q^{\omega}}A = T(A) .$$

By (6.6), $T(A)$ is not balanced projective. □

Wick defined a class of exact sequences whose projectives (called CH-projectives) include both the S-groups and the H-balanced projectives, and showed that if A is CH-projective and K an arbitrary height then $A(\mathrm{K}) = S_{\mathrm{K}}A$ is CH-projective. That is, the class of CH-projectives is closed under the functors S_{K} .

In what follows, we define an analogous class of sequences to obtain a class of groups which contains $\overline{A}$ and which is closed under both the functors S_{M} and the associated quotient functors which map A to $A/A(\mathrm{M})$.

6.8 DEFINITION. An exact sequence $0 \to B \to A \to C \to 0$ is *C-balanced* (*CH-balanced*) if both $0 \to B \to A \to C \to 0$ and the corresponding sequence $0 \to c(B) \to c(A) \to c(C) \to 0$ of cotorsion hulls are balanced (H-balanced). A group G is *C-projective* (*CH-projective*) if and only if, for every C-balanced (CH-balanced) sequence $0 \to B \to A \to C \to 0$, the natural map $\alpha^* : \mathrm{Hom}(G, A) \to \mathrm{Hom}(G, C)$ is surjective. □

Our CH-projectives coincide with those defined by Wick, although we use the term CH-balanced where he uses CH-pure. We now examine the properties of C-balanced sequences; our first result shows that the essential difference between balanced and C-balanced sequences occurs with the torsion.

6.9 THEOREM. *A balanced sequence* $0 \to B \to A \to C \to 0$ *is C-balanced if and only if the sequence*

$$0 \to T(B) \to T(A) \to T(C) \to 0$$

of torsion parts is C-balanced.

Proof. We have seen in (3.10) that the sequence $0 \to T(B) \to T(A) \to T(C) \to 0$ of torsion parts is balanced whenever $0 \to B \to A \to C \to 0$ is balanced. The groups A, B and C can be assumed reduced. We have the induced commutative diagram

$$\begin{array}{ccccccc} & & 0 & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 \to & & T(B) & \to & T(A) & \to & T(C) & \to 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 \to & & B & \to & A & \to & C & \to 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 \to & & B/T(B) & \to & A/T(A) & \to & C/T(C) & \to 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & 0 & & \end{array} \tag{15}$$

where the rows and columns are exact. Applying the functor $c(-)$ to (15) yields the following commutative diagram with exact rows and columns:

$$\begin{array}{ccccccccc}
& & 0 & & 0 & & 0 & & \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 \to & & c\big(T(B)\big) & \to & c\big(T(A)\big) & \to & c\big(T(C)\big) & \to & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 \to & & c(B) & \to & c(A) & \to & c(C) & \to & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 \to & & c\big(B/T(B)\big) & \to & c\big(A/T(A)\big) & \to & c\big(C/T(C)\big) & \to & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
& & 0 & & 0 & & 0 & &
\end{array} \tag{16}$$

The groups in the bottom row are all torsion free while those in the top row are cotorsion, so that each column of (16) splits. It follows that $0 \to c(B) \to c(A) \to c(C) \to 0$ is balanced exactly when the top and bottom rows of (16) are balanced. However, the bottom row splits and is therefore always balanced. We conclude that the balanced sequence $0 \to B \to A \to C \to 0$ is C-balanced if and only if $0 \to T(B) \to T(A) \to T(C) \to 0$ is C-balanced. □

We now show that the class of C-balanced sequences is itself closed under the functors S_{M}.

6.10 THEOREM. *If* M *is a height matrix and the sequence* $0 \to B \to A \to C \to 0$ *is* C*-balanced, then the sequence* $0 \to B(\mathsf{M}) \to A(\mathsf{M}) \to C(\mathsf{M}) \to 0$ *is* C*-balanced.*

Proof. We may assume that A, B and C are reduced. It was shown in (3.14) that the sequence $0 \to B(\mathsf{M}) \to A(\mathsf{M}) \to C(\mathsf{M}) \to 0$ is balanced. By (6.9) we can further assume that A, B and C are torsion groups. Let G be an arbitrary group. The functor $S_{\mathsf{M}} : G \mapsto G(\mathsf{M})$ is cotorsion so by Lemma 1.1 of Nunke [3] the exact sequence

$$0 \to G(\mathsf{M}) \xrightarrow{i} G \xrightarrow{\beta} G/G(\mathsf{M}) \to 0$$

induces the exact sequence

$$0 \to c\big(G(\mathsf{M})\big) \xrightarrow{i_*} [c(G)](\mathsf{M}) \xrightarrow{\beta_*} [c\big(G/G(\mathsf{M})\big)](\mathsf{M})$$

where β_* is not necessarily epic. Here, i is the inclusion map so that under i_* the torsion parts of $c\big(G(\mathsf{M})\big)$ and $[c(G)](\mathsf{M})$ are identified. We claim that $\beta_*[c(G)](\mathsf{M})$ is torsion free when G is torsion. To see this, consider the following commutative diagram, where $U = c(G(\mathsf{M}))$, $V = [c(G)](\mathsf{M})$ and $W = \beta_* V$:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
 & T(U) \arrow[r, equal] \arrow[d] & T(V) \arrow[d] & & \\
0 \arrow[r] & U \arrow[r] \arrow[d] & V \arrow[r] \arrow[d] & W \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & U/T(U) \arrow[r] \arrow[d] & V/T(V) \arrow[r] \arrow[d] & W \arrow[r] & 0 . \\
 & 0 & 0 & &
\end{tikzcd} \tag{17}$$

Now when G is torsion, $U/T(U)$ is torsion free divisible. Thus the bottom row of (17) splits and W , a summand of the torsion free group $V/T(V)$, is torsion free.

The exact sequence $0 \to B(\mathsf{M}) \to A(\mathsf{M}) \to C(\mathsf{M}) \to 0$ and the injections $i_* : c(B(\mathsf{M})) \to [c(B)](\mathsf{M})$, $j_* : c(A(\mathsf{M})) \to [c(A)](\mathsf{M})$, and $k_* : c(C(\mathsf{M})) \to [c(C)](\mathsf{M})$ give rise to the commutative diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & c(B(\mathsf{M})) \arrow[r, "i_*"] \arrow[d, "\alpha_*"] & {[c(B)](\mathsf{M})} \arrow[r] \arrow[d] & I \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & c(A(\mathsf{M})) \arrow[r, "j_*"] \arrow[d, "\beta_*"] & {[c(A)](\mathsf{M})} \arrow[r] \arrow[d] & J \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & c(C(\mathsf{M})) \arrow[r, "k_*"] \arrow[d] & {[c(C)](\mathsf{M})} \arrow[r] \arrow[d] & K \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd} \tag{18}$$

where the middle column is exact because $0 \to B \to A \to C \to 0$ is C-balanced. Since I, J and K are all torsion free, the rows in (18) all split. The middle column of (18) is balanced so the first and third columns are balanced and in particular, $0 \to B(\mathsf{M}) \to A(\mathsf{M}) \to C(\mathsf{M}) \to 0$ is C-balanced. □

Wick [1] proves that every group is the CH-balanced image of the direct sum of a group in C^Σ and an S-group, and that an exact sequence $0 \to B \to A \to C \to 0$ is CH-balanced exactly when every group in C and every group in S has the projective property relative to $0 \to B \to A \to C \to 0$. We prove the corresponding results for our situation. A preliminary lemma is required.

6.11 LEMMA. *Let* B *be a subgroup of the reduced group* A *such that* A/B *is torsion free divisible,* G *a reduced cotorsion group, and* $\phi : B \to G$ *a homomorphism. Then* ϕ *extends to a unique homomorphism* $\psi : A \to G$.

Proof. Consider the exact sequence

$$0 \to \mathrm{Hom}(A/B,\ G) \to \mathrm{Hom}(A,\ G) \to \mathrm{Hom}(B,\ G) \to \mathrm{Ext}(A/B,\ G)\ \ldots\ .$$

The conditions on A/B and G imply that $\mathrm{Hom}(A/B,\ G) = 0 = \mathrm{Ext}(A/B,\ G)$ so that $\mathrm{Hom}(A,\ G) \cong \mathrm{Hom}(B,\ G)$ under the canonical homomorphism. □

6.12 THEOREM. *An exact sequence* $0 \to B \to A \xrightarrow{\alpha} C \to 0$ *is* C*-balanced if and only if every group in* A *and every group in* S *has the projective property relative to it.*

Proof. A C-balanced sequence $0 \to B \to A \to C \to 0$ is CH-balanced and therefore every group in S has the projective property relative to it. Since $0 \to B \to A \to C \to 0$ is also balanced, (6.5) shows that every group in A has the projective property relative to it.

Conversely, suppose every group in A and in S has the projective property relative to $0 \to B \to A \to C \to 0$. By (6.5), $0 \to B \to A \to C \to 0$ is balanced, and as every group in C and every group in S has the projective property relative to it, $0 \to B \to A \to C \to 0$ is also CH-balanced. Now (6.9) allows us to assume that B, A and C are p-groups, and it is easy to see that they may also be assumed reduced. As

$0 \to c(B) \to c(A) \xrightarrow{\alpha_*} c(C) \to 0$ is H-balanced, on appealing to (3.3) we need only show that if $c \in c(C)$ and $U_p(c)$ has infinitely many gaps then there is an element a in $c(A)$ such that $\alpha_* a = c$ and $U_p(a) = U_p(c)$. Let G be a group in A containing an element g such that $U_p(g) = U_p(c)$ and $h_q(g) = \infty$ for primes $q \neq p$; as we saw in the proof of (3.10), this means that $G/T(G) \cong Q$. The correspondence $g \mapsto c$ extends to a homomorphism $\eta : G \to c(C)$ and as $T(G)$ is totally projective (see the remark following (5.28)) and $0 \to T(B) \to T(A) \to T(C) \to 0$ is balanced, the restriction η' of η to $T(G)$ extends to a $\phi : T(G) \to A$ such that $\alpha\phi = \eta'$. By (6.11) there is a homomorphism $\psi : G \to c(A)$ such that $\alpha_*\psi = \eta$. Writing $a = \psi g$ we have $U_p(a) = U_p(c)$ and $\alpha_* a = c$. □

6.13 THEOREM. *To every group* G *there is a* C*-balanced sequence* $0 \to B \to A \xrightarrow{\alpha} G \to 0$ *such that* A *is the direct sum of an* S*-group and a group in* A^{Σ} . *Thus a group is* C*-projective if and only if it is a direct summand of a direct sum of groups in* A *and a group in* S .

Proof. Let H be the direct sum of a group in C^{Σ} and an S-group such that $\psi' : H \to G$ is a $C\mathsf{H}$-balanced epimorphism. It is clear from (6.12) that $C\mathsf{H}$-balanced and C-balanced are the same in the context of torsion groups, so the proof of (6.9) shows that the restriction ψ of ψ' to $T(H)$ is a C-balanced homomorphism of $T(H)$ onto $T(G)$. By (6.3) there is a balanced epimorphism $\phi : A' \to G$ where $A' \in \mathsf{A}^{\Sigma}$. Writing $A = A' \oplus T(H)$ and $\alpha = \phi \oplus \psi$ yields the result. □

We also have the analogue of (6.6):

6.14 PROPOSITION. *(i) The torsion part of a* C*-projective (*$C\mathsf{H}$*-projective) is in* $\overline{S}$.

(ii) A torsion (torsion free) group is C*-projective if and only if it is in* $\overline{S}$ *(completely decomposable); the same holds for* $C\mathsf{H}$*-projective.*

(iii) If A *is* C*-projective (*$C\mathsf{H}$*-projective) then* $A/T(A)$ *is completely decomposable.*

(iv) Every torsion summand of a C*-projective is in* $\overline{S}$. □

The following three lemmas will be used, often without explicit mention, in our next theorem. The proofs of the first two are trivial.

6.15 LEMMA. *Let* A *and* B *have torsion free rank* 1 *and let* $B \leq A$. *If* $B_p = A_p$ *then* $(A/B)_p \leq Z(p^\infty)$. □

6.16 LEMMA. *For any group* A *and height matrix* M *we have*

$$[A(\mathsf{M})]_p = A_p(\mathsf{M}) = A_p(\mathsf{M}_p) = [A(\mathsf{M}_p)]_p \text{ .} \quad \square$$

6.17 PROPOSITION. *Let* $0 \to B \to A \to C \to 0$ *be an exact sequence of p-groups. If* $B \leq Z(p^\infty)$ $(C \leq Z(p^\infty))$ *and* C (B) *is totally projective and reduced then* A *is totally projective.*

Proof. When $B \leq Z(p^\infty)$ the result is trivial. The case when $C \leq Z(p^\infty)$ and C is finite is also straightforward. In the remaining case, $A/B \cong Z(p^\infty)$ and $A = \bigcup_{n<\omega} A_n$ where $A_n/B \cong Z(p^n)$. Let $0 \to U \to V \xrightarrow{\alpha} W \to 0$ be an arbitrary balanced sequence of torsion groups and $\phi : A \to W$ a homomorphism. It suffices to show that ϕ lifts to a $\psi : A \to V$ such that $\alpha\psi = \phi$. Writing ϕ_n for the restriction of ϕ to A_n we use (6.1) to lift each ϕ_n to a $\psi_n : A_n \to V$ such that, for each $m < n$, the restriction of ψ_n to A_m is ψ_m . Clearly $\psi = \bigcup_{n<\omega} \psi_n$ is of the required form. □

6.18 THEOREM. *Let* A *be a C-projective group and let* M *be a height matrix. Then both* $A(\mathsf{M})$ *and* $A/A(\mathsf{M})$ *are C-projective.*

Proof. It follows from (6.13) that we need only consider A to be a reduced group in A , or a reduced S-group.

Suppose $A \in \mathsf{A}$. By (2.19) we can assume A has torsion free rank 1 . On several occasions, we will prove that a group G with torsion free rank 1 is a member of A - to do this it suffices to find, for each prime p , an element g_p in G with infinite order such that $(G/\langle g_p \rangle)_p$ is a totally projective p-group. This will be automatic from (2.26) whenever G_p is totally projective. We begin with $A(\mathsf{M})$ and distinguish two cases.

CASE I. $A(\mathfrak{M})$ is torsion. Then

$$A(\mathfrak{M}) = A(\mathfrak{M}) \cap T(A) = [T(A)](\mathfrak{M}) .$$

Since $T(A)$ is an S-group, (2.20) shows that $A(\mathfrak{M})$ is an S-group.

CASE II. $A(\mathfrak{M})$ has torsion free rank 1. We show $A(\mathfrak{M}) \in \mathcal{A}$. When A_p is totally projective, so is $A_p(\mathfrak{M}) = [A(\mathfrak{M})]_p$. When A_p is not totally projective, it is an S-group, and the proof of (5.28) shows that A contains an element a of infinite order with $U_p(a) = (\lambda, \lambda+1, \lambda+2, \ldots)$ for some limit ordinal λ not cofinal with ω. The proof of (5.29) reveals that we can split off a torsion summand from A in such a way that the p-component of the complement has p-length λ. From this and (5.10) it is clear that we may assume that A_p is λ-elementary.

Write $\mathfrak{M}_p = (\sigma_0, \sigma_1, \ldots)$. If some $\sigma_i \geq \lambda$ then $A_p(\mathfrak{M})$ is bounded and therefore totally projective. The remaining possibility is $\sigma_i < \lambda$ for $i = 0, 1, \ldots$. It is now possible to arrange that $a \in A(\mathfrak{M})$. Setting $\gamma = \mathfrak{M}_p * \omega = \lim_{i<\omega} \sigma_i$, (2.8) and (2.12) yield $p^\omega[A(\mathfrak{M})] = p^\gamma A \cap A(\mathfrak{M})$. As $\sigma_i < \lambda$ for all i and γ is cofinal with ω, it follows that $\gamma < \lambda$ and that $a \in p^\omega[A(\mathfrak{M})]$. This shows that a has infinite p-height in $A(\mathfrak{M})$. Now $Z_p \otimes A(\mathfrak{M})$ is a subgroup of $(Z_p \otimes A)(\mathfrak{M})$ containing $1 \otimes a$ and with the same torsion part. Since $1 \otimes a$ has infinite p-height in $Z_p \otimes A(\mathfrak{M})$, we have $Z_p \otimes A(\mathfrak{M}) = (Z_p \otimes A)(\mathfrak{M})$. Thus

$$\begin{aligned}
(A(\mathfrak{M})/\langle a\rangle)_p &\cong ((Z_p \otimes A(\mathfrak{M}))/\langle 1 \otimes a\rangle)_p \\
&= ((Z_p \otimes A)(\mathfrak{M})/\langle 1 \otimes a\rangle)_p \\
&= (Z_p \otimes A/\langle 1 \otimes a\rangle)_p(\mathfrak{M}) \quad \text{by (2.6)} \\
&\cong (A/\langle a\rangle)_p(\mathfrak{M}) .
\end{aligned}$$

As $A \in \mathcal{A}$ we have $(A/\langle a\rangle)_p$, and therefore $(A/\langle a\rangle)_p(\mathfrak{M})$, totally projective. It follows that $A(\mathfrak{M}) \in \mathcal{A}$.

We turn now to $A/A(\mathfrak{M})$ and begin by showing that $[A/A(\mathfrak{M})]_p$ is totally projective whenever A_p is totally projective. In the exact sequence

$$0 \to A(\mathfrak{M})/T(A(\mathfrak{M})) \to A/T(A(\mathfrak{M})) \xrightarrow{\eta} A/A(\mathfrak{M}) \to 0$$

the first term is torsion free, while the middle term has torsion free rank 1. Arguing as in (5.3) we have $T(A/A(\mathfrak{M}))/\eta(T(A)/T(A(\mathfrak{M})))$ locally cyclic. For the p-components,

$$0 \to A_p/A_p(\mathfrak{M}) \xrightarrow{\eta'} [A/A(\mathfrak{M})]_p \to Z(p^\alpha) \to 0$$

is exact where $\alpha \in \{\infty\} \cup \{0, 1, \ldots\}$ and η' is the restriction of η to $A_p/A_p(\mathfrak{M})$. By (2.6), $A_p/A_p(\mathfrak{M})$ is reduced and (6.17) shows $[A/A(\mathfrak{M})]_p$ is totally projective. Thus we are again reduced to considering what happens when A_p is a λ-elementary S-group for some limit ordinal λ not cofinal with ω. The same two cases arise as for $A(\mathfrak{M})$.

CASE I. $A/A(\mathfrak{M})$ is torsion. We will prove that $A/A(\mathfrak{M})$ is totally projective. Write $\mathfrak{M}_p = (\sigma_0, \sigma_1, \ldots)$. Since $A/A(\mathfrak{M})$ is torsion and since A has an element a of infinite order such that $U_p(a) = (\lambda, \lambda+1, \lambda+2, \ldots)$, we must have $\sigma_i < \lambda + \omega$ for all i, and if $\sigma_i \geq \lambda$ for some i then $\mathfrak{M}_p$ can have only finitely many gaps. Replacing a, if necessary, by a suitable p-power multiple of itself we may therefore assume $h_p^A(a) \geq \tau(\mathfrak{M}_p)$. Writing $B/\langle a \rangle = (A/\langle a \rangle)(\mathfrak{M})$, we have

$$\begin{aligned}(B/\langle a \rangle)_p &= (A/\langle a \rangle)_p(\mathfrak{M}) \\ &= (A(\mathfrak{M}_p)/\langle a \rangle)_p \quad \text{by (2.6)}\end{aligned}$$

so that $B_p = A(\mathfrak{M})_p$ and (6.15) yields $(B/A(\mathfrak{M}))_p \leq Z(p^\infty)$. In the exact sequence

$$0 \to (B/A(\mathfrak{M}))_p \to (A/A(\mathfrak{M}))_p \to (A/\langle a \rangle)_p/(A/\langle a \rangle)_p(\mathfrak{M}) \to 0$$

the last term is totally projective and reduced, so that $(A/A(\mathfrak{M}))_p$ is totally projective.

CASE II. $A/A(\mathsf{M})$ has torsion free rank 1. In the case when all the entries of $\mathsf{M}_p = (\sigma_0, \sigma_1, \ldots)$ are less than λ, we set $\tau = \tau(\mathsf{M}_p)$. Then

$$(A/A(\mathsf{M}))_p \cong A_p/A_p(\mathsf{M})$$
$$\cong \left(A_p/p^{\tau}A_p\right)/\left(A_p/p^{\tau}A_p\right)(\mathsf{M})$$

is totally projective since $A_p/p^{\tau}A_p$ is totally projective. Suppose, on the other hand, that $\sigma_i \geq \lambda$ for some i. There exists an integer $n \geq 0$ such that $p^n[A(\mathsf{M})] = A(p^n\mathsf{M})$ has zero p-component. Taking any a in A of infinite order, the isomorphism

$$\left(A/\left(A(p^n\mathsf{M}) \oplus \langle a\rangle\right)\right)_p \cong (A/\langle a\rangle)_p \tag{19}$$

shows that the left hand side is totally projective. As before, we can arrange that $h_p^A(a) \geq \lambda$. Let $C = A(p^n\mathsf{M}) \oplus \langle a\rangle$ and $B/C = (A/C)(\mathsf{M})$. We claim $B_p = [A(\mathsf{M})]_p$. For if $b \in B_p$ we have $h_p^A(p^nb) < \lambda$ for all $n \geq 0$ such that $p^nb \neq 0$. Now $C \leq p^{\lambda}A$ so by (2.5), $h_p^{A/C}(p^nb+C) = h_p^A(p^nb)$ for all such n and therefore $b \in A_p(\mathsf{M})$. The inclusion $A_p(\mathsf{M}) \leq B_p$ is trivial. By (6.15), $\left[B/\left(A(\mathsf{M}) \oplus \langle a\rangle\right)\right]_p \leq Z(p^{\infty})$. The sequence

$$0 \to \left(B/\left(A(\mathsf{M}) \oplus \langle a\rangle\right)\right)_p \to \left(A/\left(A(\mathsf{M}) \oplus \langle a\rangle\right)\right)_p \to (A/B)_p \to 0$$

is exact and since $(A/B)_p \cong (A/C)_p/(A/C)_p(\mathsf{M})$ is totally projective and reduced (see (19) above), $\left(A/\left(A(\mathsf{M}) \oplus \langle a\rangle\right)\right)_p$ is totally projective. Now

$$\left(A/\left(A(\mathsf{M}) \oplus \langle a\rangle\right)\right)_p \cong \left[\left(A/A(\mathsf{M})\right)/\langle a+A(\mathsf{M})\rangle\right]_p$$

and $a + A(\mathsf{M})$ has infinite order in $A/A(\mathsf{M})$, so $A/A(\mathsf{M}) \in \mathsf{A}$.

Finally, suppose A is an S-group. We can assume that A is a p-group and that A is λ-elementary for some ordinal λ not cofinal with ω. By (2.20), $A(\mathsf{M})$ is an S-group and it remains only to consider $A/A(\mathsf{M})$. Write $\mathsf{M}_p = (\sigma_0, \sigma_1, \ldots)$. In the case when $\sigma_i < \lambda$ for all

i , putting $\tau = \tau(\mathsf{M}_p)$ gives

$$A/A(\mathsf{M}) = A/A(\mathsf{M}_p)$$

$$\cong (A/p^{\tau}A)/(A/p^{\tau}A)(\mathsf{M}) .$$

As $A/p^{\tau}A$ is totally projective, so is $A/A(\mathsf{M})$. When $\sigma_i \geq \lambda$ for some i , let n be the greatest integer for which $\sigma_n < \lambda$ (if $\sigma_i \geq \lambda$ for all i , there is nothing to prove). Let H be a totally projective p-group which imbeds A as a λ-dense isotype subgroup such that $H/A \cong Z(p^{\infty})$; clearly $\mathsf{u} = (\lambda, \lambda+1, \ldots)$ and H are compatible, so let G be a group in A represented by (u, H) . By (5.10), $T(G)$ is isomorphic to a λ-dense, isotype subgroup of H with corank 1 and lemma 3.1 of Warfield [2] shows that $A \cong T(G)$. Now let $\mathsf{v} = (\sigma_0, \sigma_1, \ldots, \sigma_n, \infty, \ldots)$ so that $A(\mathsf{v}) = A(\mathsf{M})$ and $A/A(\mathsf{M}) \cong T(G/G(\mathsf{v}))$ because $G(\mathsf{v})$ is torsion. As $G/G(\mathsf{v}) \in \mathsf{A}$, it follows that $T(G/G(\mathsf{v}))$ is an S-group. □

The following result (which was promised in Chapter 2) follows directly from (2.20) and (6.18).

6.19 PROPOSITION. *If* $S \in \mathsf{S}$ *and* u *is an arbitrary indicator then* $S(\mathsf{u})$ *and* $S/S(\mathsf{u})$ *are both in* S . □

Now for the main result connecting the class of C-projectives with the class of balanced projectives.

6.20 THEOREM. *Let* M *be the closure of the class of balanced projectives under the functors* S_{M} . *Then* $\overline{\mathsf{M}}$ *is the class of* C*-projectives.*

Proof. As every balanced projective is C-projective, (6.18) shows that M , and therefore $\overline{\mathsf{M}}$, is contained in the class of C-projectives.

For the converse, it suffices to show that every λ-elementary S-group A is of the form $G(\mathsf{M}) = S_{\mathsf{M}}G$ for some group G in A and some height matrix M . Let H be a totally projective p-group which imbeds A as a

λ-dense, isotype subgroup and let G be a group in A represented by (u, H), where $u = (\lambda, \lambda+1, \lambda+2, \ldots)$. It was shown in the proof of (6.18) that $T(G) = A$. However, since the q-height of every element of infinite order in G is finite when $q \neq p$, it follows that $A = q^{\omega}G$. □

CHAPTER 7

INVARIANTS AND CLASSIFICATION THEOREMS

One of the classes of groups investigated in the earlier chapters was the class of H-projectives. It is therefore pleasing to be able to report that the isomorphism classifications of completely decomposable torsion free groups and of totally projective groups by means of complete sets of invariants have a common generalisation to this class, and indeed can have no further generalisation. The first half of this chapter reports on this and the second half discusses some of the difficulties which prevent the direct extension of these results to balanced projectives and gives a partial result indicating a possible alternative.

H-projectives

In this section we give invariants and a classification theorem for groups in C^{Σ}. A result of Warfield [4] is given which shows that $\overline{C} = C^{\Sigma}$, thus yielding a complete classification of the H-projectives by invariants.

First we define groups which give our invariants a more general setting.

7.1 DEFINITION. A group in which the height matrix of every element has finitely many gaps is called *gap-finite*. □

Observe that torsion, torsion free and $\overline{C}$ groups are all gap-finite, and direct sums and direct summands of gap-finite groups are gap-finite.

7.2 DEFINITION. Call the equivalence class $\tilde{M}$ of a height matrix M having finitely many gaps a *type*. If t_1 and t_2 are types, then $t_1 < t_2$ if $t_1 \neq t_2$ and there are representative height matrices H_1 in t_1, H_2 in t_2 such that $H_1 < H_2$.

Given a gap-finite group A and type $\mathbf{t}$, we define subgroups $A(\mathbf{t})$ and $A^*(\mathbf{t})$ as follows:

$$A(\mathbf{t}) = \{a \in A : \tilde{\mathsf{H}}(a) \geq \mathbf{t}\}$$

and $A^*(\mathbf{t})$ is the subgroup generated by

$$\{a \in A : \tilde{\mathsf{H}}(a) > \mathbf{t}\} . \qquad \square$$

It is readily shown that the given partial order on types is well-defined, and is compatible with the multiplication of height matrices by positive integers defined in Chapter 2.

Let B be a summand of a gap-finite group A and $\mathbf{t}$ an arbitrary type. Then $B(\mathbf{t}) = A(\mathbf{t}) \cap B$ and $B^*(\mathbf{t}) = A^*(\mathbf{t}) \cap B$. Given a direct sum $A = \oplus A_i$, it is easy to see that $A(\mathbf{t}) = \oplus A_i(\mathbf{t})$ and $A^*(\mathbf{t}) = \oplus A_i^*(\mathbf{t})$. Since torsion elements have the same type as elements in a divisible group, $T(A) \leq A^*(\mathbf{t}) \leq A(\mathbf{t})$ for every gap-finite group A and type $\mathbf{t}$. If $\tilde{\mathsf{H}}(a) = \mathbf{t}$ for some a in A, we say the type $\mathbf{t}$ *occurs* in A.

7.3 DEFINITION. Let A be a gap finite group of torsion free-rank at most 1, and let a be an element of A such that $A/\langle a \rangle$ is torsion. Then we define the *type*, $\mathbf{t}(A)$, *of* A to be $\tilde{\mathsf{H}}(a)$. $\square$

The following proposition generalises a result of Baer.

7.4 PROPOSITION. *Let* $C = \bigoplus_{i \in I} C_i = \bigoplus_{j \in J} B_j$ *where the* C_i *and the* B_j *are all gap-finite groups of torsion free rank* 1. *Then there is a bijection* $\phi : I \to J$ *such that* C_i *and* $B_{\phi(i)}$ *have the same type.*

Proof. Writing $J_{\mathbf{t}} = \{i \in I : \mathbf{t}(C_i) = \mathbf{t}\}$ for those types $\mathbf{t}$ occurring in C, it follows that

$$C(\mathbf{t})/C^*(\mathbf{t}) \cong \bigoplus_{J_{\mathbf{t}}} \left(C_i(\mathbf{t})/C_i^*(\mathbf{t})\right)$$

is a direct sum of torsion free groups of rank 1, with exactly one summand corresponding to each i in $J_{\mathbf{t}}$. The rank of $C(\mathbf{t})/C^*(\mathbf{t})$ is manifestly an invariant of C so that $|J_t|$ is independent of the decomposition of C into groups having torsion free rank 1 . □

Let A be a direct sum of gap-finite groups of torsion free rank 1 and suppose $A = B \oplus C$. Since

$$A(\mathrm{t})/A^*(\mathrm{t}) \cong B(\mathrm{t})/B^*(\mathrm{t}) \oplus C(\mathrm{t})/C^*(\mathrm{t})$$

it follows that $B(\mathrm{t})/B^*(\mathrm{t})$ is completely decomposable because $A(\mathrm{t})/A^*(\mathrm{t})$ is. In the light of this, we define:

7.5 DEFINITION. Let $A = \oplus A_i$, where each A_i is gap-finite and has torsion free rank 1 . For each t occurring in A , let $k(A, \mathrm{t})$ denote the number of summands in any decomposition of $A(\mathrm{t})/A^*(\mathrm{t})$ into torsion free groups of rank 1 . □

Now the classification theorem for groups in C^Σ can be proved.

7.6 THEOREM. *Let B and C be reduced groups in C^Σ . Then $B \cong C$ if and only if $k(B, \mathrm{t}) = k(C, \mathrm{t})$ for all types t and $f_\sigma^{\,p}(B) = f_\sigma^{\,p}(C)$ for all primes p and ordinals σ .*

Proof. Only half the statement requires proof. Let $B = \bigoplus_{i \in I} B_i$ and $C = \bigoplus_{j \in J} C_j$, where the B_i and C_j are groups in C . There is a bijection $\phi : I \to J$ such that $\mathrm{t}(B_i) = \mathrm{t}(C_{\phi(i)})$ for all i in I ; accordingly we identify I with J . For each i choose b_i in B_i and c_i in C_i such that $B_i/\langle b_i \rangle$ and $C_i/\langle c_i \rangle$ are torsion. We can also arrange that $\mathrm{H}(b_i) = \mathrm{H}(c_i)$ for all i , and that these height matrices contain no gaps. The correspondences $b_i \longmapsto c_i$ give rise to a height preserving isomorphism $\oplus \langle b_i \rangle \to \oplus \langle c_i \rangle$. Write $G = \oplus \langle b_i \rangle$ and $H = \oplus \langle c_i \rangle$, and let B^p $(B_i^{\,p})$ be the full inverse image of $(B/G)_p$

$\left((B_i/\langle b_i\rangle)_p\right)$ under the natural homomorphism $B \to B/G$ $\left(B_i \to B_i/\langle b_i\rangle\right)$. As $\langle b_i\rangle$ is p-nice in B_i^p for each i in I (see (2.24)), G is p-nice in B^p . The same holds for H and C^p . Although there is little hope for $f_\sigma^{\,p}\left(B_i^{\,p}, \langle b_i\rangle\right) = f_\sigma^{\,p}\left(C_i^{\,p}, \langle c_i\rangle\right)$ individually, we have

$$f_\sigma^{\,p}(B^p, G) = \sum_{i \in I} f_\sigma^{\,p}\left(B_i^{\,p}, \langle b_i\rangle\right)$$

$$= \sum_{i \in I} f_\sigma^{\,p}\left(B_i^{\,p}\right) \quad \text{by (5.6)}$$

$$= f_\sigma^{\,p}(B)$$

$$= f_\sigma^{\,p}(C^p, H) \ .$$

Applying (2.14) for each prime p and then (2.21), we have $B \cong C$. □

Warfield has made considerable use of a category (which he denotes by H) whose objects are groups and whose morphisms are defined in terms of height preserving homomorphisms between certain subgroups. We shall require only the concept of H-isomorphism which is defined as follows:

7.7 DEFINITION (Warfield [1]). Groups A and B are said to be *H-isomorphic* if there are subgroups G and H of A and B respectively such that

(i) A/G and B/H are torsion; and

(ii) there is a height preserving isomorphism $\alpha : G \to H$ (where heights are taken in A and B). □

We end this section with a brief account of Warfield's proof that $\overline{C} = C^\Sigma$ (see Warfield [4]). He begins by showing that every summand A of a direct sum of gap-finite groups having torsion free rank 1 is H-isomorphic to a direct sum B of gap-finite groups of torsion free rank 1 ; further, B can be chosen to have the same Ulm invariants as A . The task is then reduced to modules over a complete discrete valuation ring using two lemmas, one corresponding to (2.21) and the other showing that certain height preserving isomorphisms between submodules extend to the modules themselves provided this happens for their completions. The proof is completed using a version of Hill's proof of Ulm's theorem for totally projective groups - the completeness of the valuation ring is essential for every finitely generated submodule of a module over such a ring to be p-nice. We state Warfield's result in:

7.8 THEOREM (Warfield [4]). *Every* H*-balanced projective is a direct sum of groups in* C *. An* H*-balanced projective* C *is determined up to isomorphism by its Ulm invariants and the invariants* $k(C, \mathsf{t})$ *for all types* t *. If* $\mathcal{D}$ *is a class of groups such that*

(i) $\mathcal{D}$ *is closed under the formation of direct sums;*

(ii) $\mathsf{C}^{\Sigma} \subseteq \mathcal{D}$ *;*

(iii) the groups in $\mathcal{D}$ *are classified up to isomorphism by the same invariants as those for* $\overline{\mathsf{C}}$ *;*

then $\overline{\mathsf{C}} = \mathcal{D}$. □

Balanced projectives

In this section we discuss the problem of finding invariants for, and classifying, the balanced projectives. One immediate snag is the lack of a natural partial ordering of height matrix equivalence classes. Indeed, this difficulty is already present with indicators as can be seen from the following. Let $\mathsf{u} = (0, 2, 4, \ldots)$ and $\mathsf{v} = (1, 3, 5, \ldots)$ be two p-indicators, where p is some (fixed) prime. Then $\mathsf{u} < \mathsf{v}$ and $\mathsf{v} < p\mathsf{u}$ but $\tilde{\mathsf{u}} \neq \tilde{\mathsf{v}}$. This failure of anti-symmetry shows that the partial order defined on indicators does not induce a partial order on their equivalence classes. (It is clear from Chapter 5 that there exist groups in A having elements with the above indicators.) Clearly we cannot proceed using the methods of the previous section.

Warfield [1] was able to overcome this difficulty for modules over Z_p (recall that these are just Abelian groups which are q-divisible for all primes $q \neq p$). He showed that every balanced projective Z_p-module (which he called a T^*-module) is $\mathcal{H}$-isomorphic to a direct sum of T^*-modules having torsion free rank 1 , and that the equivalence classes of the p-indicators associated with the members of such a direct sum are invariant for every T^*-module. This result cannot be strengthened to show that every T^*-module is a direct sum of T^*-modules of torsion free rank 1 , as is shown by an example of Rotman and Yen [1]. Thus, although T^*-modules are not always direct sums of the basic building blocks (the T^*-modules with torsion free rank 1), they do have what is known as a decomposition basis - a concept first introduced by Rotman [1]:

7.9 DEFINITION (Rotman [1]). Let A be a group. A subset $\{x_i\}_{i\in I}$ of A is *a decomposition basis* of A if

(i) the x_i's are independent and generate a maximal free subgroup of A ; and

(ii) for each prime p , elements $x_1, \ldots, x_n$ in $\{x_i\}$ and integers $r_1, \ldots, r_n$ we have $h_p(r_1x_1 + \ldots + r_nx_n) = \min\limits_{1\leq i\leq n} h_p(r_ix_i)$. □

It is clear that a group has a decomposition basis if and only if it is $\mathcal{H}$-isomorphic to a direct sum of groups in $\mathcal{A}$. We require the following definition which extends the notion of equivalence for height matrices.

7.10 DEFINITION. For two sets S and S' of height matrices we say that S is *equivalent* to S' (in symbols, $S \sim S'$) if there is a bijection $\phi : S \to S'$ such that $\mathbb{M} \sim \phi(\mathbb{M})$ for all $\mathbb{M}$ in S . If X is a subset of A then we denote the set of height matrices (taken in A) of elements in X by $\mathbb{H}(X)$, that is,

$$\mathbb{H}(X) = \{\mathbb{H}(x) : x \in X\} . \quad \square$$

We turn our attention to groups having a decomposition basis. It is natural to ask whether $\mathbb{H}(X) \sim \mathbb{H}(Y)$ whenever X and Y are decomposition bases of the same group. The answer is negative, even for groups in Λ^{Σ}. We illustrate this with an example similar to that of Warfield [3]. Since (as we have already mentioned) this holds for T^*-modules, difficulties can only arise for groups which are not divisible at more than one prime.

7.11 EXAMPLE. Let $\mathsf{u} = (0, 2, 4, 6, \ldots)$, $\mathsf{v} = (1, 3, 5, 7, \ldots)$ and $\mathsf{w} = (0, 1, 2, 3, \ldots)$. Let A_1, A_2 be groups of torsion free rank 1 with elements a_1 in A_1 and a_2 in A_2 of infinite order such that $U_2(a_1) = \mathsf{u}$, $U_2(a_2) = \mathsf{v}$, $U_3(a_1) = \mathsf{v}$, $U_3(a_2) = \mathsf{u}$ and $U_p(a_1) = U_p(a_2) = \mathsf{w}$ when $p \neq 2, 3$. Then in $A = A_1 \oplus A_2$ we set $a_3 = a_1 + a_2$ and $a_4 = 2a_1 + 3a_2$ so that $U_2(a_3) = \mathsf{u}$, $U_2(a_4) = \mathsf{v}$, $U_3(a_3) = \mathsf{u}$, $U_3(a_4) = \mathsf{v}$, and $U_p(a_3) = U_p(a_4) = \mathsf{w}$ when $p \neq 2, 3$. We claim that $\{a_3, a_4\}$ is a decomposition basis for A. Let $x = na_3 + ma_4$ be any linear combination of a_3 and a_4. Since the 2- and 3-heights of multiples of a_3 and a_4 can never coincide, we have

$$h_p(x) = \min\{h_p(na_3), h_p(ma_4)\} \quad \text{when } p = 2, 3 .$$

The remaining case to check is when $p \neq 2, 3$ and $h_p(na_3) = h_p(ma_4) = k$. Clearly p^k divides both n and m and no heigher power of p divides either. Thus

$$\begin{aligned} k \le h_p(x) &= h_p(na_3+ma_4) \\ &= \min\{h_p((n+2m)a_1), h_p((n+3m)a_2)\} \\ &= l . \end{aligned}$$

Since $\{a_1, a_2\}$ is a decomposition basis for A then $l > k$ implies p^{k+1} divides both $n + 2m$ and $n + 3m$. But then p^{k+1} divides $(n+3m) - (n+2m) = m$, a contradiction. We conclude that

$k = l = \min\{h_p(na_3), h_p(ma_3)\}$ and $\{a_3, a_4\}$ is another decomposition basis for A. Clearly $\mathsf{H}(\{a_1, a_2\}) \nsim \mathsf{H}(\{a_3, a_4\})$ and we have our counterexample. □

Note that choosing $W = (\infty, \infty, \ldots)$ avoids the second case in the above argument. However, Rotman has shown that every countable group A with finite torsion free rank and with every element of finite order having finite p-height for all p, decomposes as a direct sum of groups with torsion free rank 1 whenever A has a decomposition basis. Further, the decomposition can be chosen so that each element of such a basis lies in a different summand of the decomposition. With this in mind, the impact of (7.11) can be strengthened by observing that when A is countable,

$$A = A_1 \oplus A_2 = A_3 \oplus A_4 \text{ , where } a_i \in A_i \text{ for } i = 1, \ldots, 4 .$$

Bypassing the difficulty presented by (7.11), Warfield [3] has shown:

7.12 THEOREM (Warfield [3]). *Let* A *and* B *be groups in* A^{Σ} *and suppose* $A = \bigoplus_{i \in I} A_i$ *and* $B = \bigoplus_{j \in J} B_j$ *where each* A_i *and* B_j *has torsion free rank* 1. *Suppose we can find a bijection* $\phi : I \to J$ *such that for each* i *in* I *there are elements* a_i *and* b_i *in* A_i *and* $B_{\phi(i)}$, *respectively, such that* $\mathsf{H}(a_i) \sim \mathsf{H}(b_i)$ *and suppose that* A *and* B *have the same Ulm invariants. Then* $A \cong B$. □

This result was proved by Warfield for simply presented groups but the proof is equally valid for groups in the larger class A^{Σ}.

The example in (7.11) suggests the following definition of equivalence between sets of height matrices:

7.13 DEFINITION. Let S and S' be two sets of height matrices. If A is a group having decomposition bases X and Y for which $\mathsf{H}(X) \sim S$ and $\mathsf{H}(Y) \sim S'$ then we say that S and S' are *basis-equivalent* with respect to A - we write

$$S \overset{A}{\otimes} S' . \quad \square$$

The following result on basis-equivalence shows that the reference to a particular group A may be dropped, and consequently that $\otimes$ is a well-defined equivalence relation.

7.14 LEMMA. *Let S and S' be two sets of height matrices and suppose that $S \overset{A}{\otimes} S'$ for some group A. If B is another group with a decomposition basis V for which $\mathbb{H}(V) \sim S$ then B has a decomposition basis W such that $\mathbb{H}(W) \sim S'$.*

Proof. Suppose $S = \mathbb{H}(X)$ and $S' = \mathbb{H}(Y)$ for decomposition bases X and Y of A. Let $\phi : X \to V$ be a bijection, and denote by ϕ^* the natural extension of ϕ to an isomorphism $\langle X \rangle \to \langle V \rangle$. Replacing, if necessary, elements of X, Y and V with suitable multiples of themselves we may assume that $y \in \langle X \rangle$ for all y in Y and $\mathbb{H}(x) = \mathbb{H}(\phi x)$ for all x in X. Define a map $\psi : Y \to B$ by $\psi y = \phi^* y$. Clearly ψ is injective, $W = \psi Y$ is an independent set of torsion free elements of B and B/W is torsion. Let ψ^* denote the corresponding isomorphism $\psi^* : \langle Y \rangle \to \langle W \rangle$. We show that W is the required decomposition basis. Since ϕ^* leaves heights unaltered (where heights are taken with respect to A and B), the same applies to $\psi^* = \phi^*|_{\langle Y \rangle}$. Thus, W is a decomposition basis for B and $\mathbb{H}(y) = \mathbb{H}(\psi y)$ for all y in Y. $\square$

I have so far been unable to determine whether an arbitrary balanced projective group A has a decomposition basis, and therefore whether the $\otimes$ equivalence classes are invariants applicable to every such A; it is a consequence of Warfield's work, however, that $Z_p \otimes A$ has a decomposition basis for each prime p.

CHAPTER 8

BALANCED EXTENSIONS

In this chapter we explore the properties of $\mathrm{Bext}(C, A)$ for groups A and C. We introduce the functor Bext which is naturally associated with $\mathrm{Bext}(C, A)$ and indicate some of its homological properties. Various classes of groups for which $\mathrm{Bext}(C, A) = 0$ are then characterised. In particular, $\mathrm{Bext}(C, A) = 0$ for all torsion free groups C if and only if A is cotorsion. On the other hand, the groups C for which $\mathrm{Bext}(C, A) = 0$ for all torsion free groups A are characterised as homomorphic images of groups in A^{Σ} where the homomorphisms have torsion kernel, or equivalently as groups C for which $C/T(C)$ is completely decomposable. It turns out that every group with a decomposition basis, and every summand of a direct sum of mixed groups with torsion free rank 1 has this property. We show that a group A satisfies $\mathrm{Bext}(C, A) = 0$ for all torsion groups C exactly when $T(A)$ is torsion complete. We conclude with some results on groups which can be represented as the balanced extension of a torsion group by a torsion free group - that is, groups with balanced torsion part.

It was noted in Chapter 3 that if

$$e : 0 \to A \to G \to C \to 0$$

is an element of $\mathrm{Bext}(C, A)$, then the sequence

$$0 \to A_D \to G_D \to C_D \to 0$$

of maximal divisible subgroups is exact, and therefore splits off. Thus $\mathrm{Bext}(C, A) = 0$ whenever A or C is divisible and we need only consider the case where both A and C are reduced.

Let us examine the relationship between $\mathrm{Bext}(C, A)$ and other subgroups of $\mathrm{Ext}(C, A)$. Every balanced subgroup is pure, so that

$$\mathrm{Bext}(C, A) \leq \mathrm{Pext}(C, A) \tag{20}$$

for all groups A and C. This inclusion is in general proper as there are many examples of groups A for which $\mathrm{Pext}(Q, A)$ and $\mathrm{Pext}(Z(p^\infty), A)$ are non-zero. However, by (E) of p.42 in Chapter 3, we have equality in (20) whenever C is torsion and $p^\omega C = 0$ for each prime p. Noting that $\mathrm{Pext}(C, A) = \mathrm{Ext}(C, A)(K)$ where $K = \langle\langle \omega, \omega, \ldots \rangle\rangle$ it will be seen that the following proposition generalises (E) in Chapter 3.

8.1 PROPOSITION. *Let* C *be a reduced torsion group,* K *a height such that* $C(K) = 0$. *Then* $\mathrm{Ext}(C, A)(K) \leq \mathrm{Bext}(C, A)$ *for every* A.

Proof. Let $K = \langle\langle \beta_2, \beta_3, \ldots, \beta_p, \ldots \rangle\rangle$. If $e : 0 \to A \to G \to C \to 0$ is an element of $\mathrm{Ext}(C, A)(K)$ then $e \in p^{\beta_p}\mathrm{Ext}(C, A)$ for all p in P. By Lemma 1.10 of Nunke [3],

$$p^\beta C[p] = (A + p^\beta G[p])/A \tag{21}$$

for all $\beta < \beta_p$ and p in P. Now $p^{\beta_p} C[p] = 0$ for each p, so (21) holds trivially for $\beta \geq \beta_p$. Hence $T(A)$ is balanced in $T(G)$ by (b′) on p. 34 of Chapter 3. Now (3.11) shows $e \in \mathrm{Bext}(C, A)$. □

Given a height matrix M, let $K = \langle\langle \beta_2, \beta_3, \ldots, \beta_p, \ldots \rangle\rangle$ be the height formed by taking the first column of M. For every torsion group C we have $C(M) = 0$ if and only if $C(K) = 0$. Since $\mathrm{Ext}(C, A)(M) \leq \mathrm{Ext}(C, A)(K)$ it follows that (8.1) holds for K replaced by M, but that no sharpening of the result is obtained this way.

We now turn to the functor Bext. In Chapter 3 it was shown that the balanced exact sequences form a proper class and therefore determine a subfunctor Bext of Ext in the manner described in Mac Lane [1]. In particular, given homomorphisms $\alpha : A \to A'$ and $\gamma : C' \to C$, the restriction of the natural map $\mathrm{Ext}(\gamma, \alpha) : \mathrm{Ext}(C, A) \to \mathrm{Ext}(C', A')$ to $\mathrm{Bext}(C, A)$ yields the homomorphism

$$\mathrm{Bext}(\gamma, \alpha) : \mathrm{Bext}(C, A) \to \mathrm{Bext}(C', A') .$$

Let Ab be the category of all Abelian groups. In categorical language Bext is an additive bifunctor on $Ab \times Ab$, contravariant in the first variable and covariant in the second. For brevity we shall denote both $\mathrm{Ext}(\gamma, 1_A)$ and $\mathrm{Bext}(\gamma, 1_A)$ by γ^*, similarly α_* denotes both $\mathrm{Ext}(1_{C'}, \alpha)$ and its restriction $\mathrm{Bext}(1_{C'}, \alpha)$. With each balanced extension

$$0 \to U \xrightarrow{\alpha} V \xrightarrow{\beta} W \to 0$$

and group A are associated the sequences

$$0 \to \mathrm{Hom}(A, U) \to \mathrm{Hom}(A, V) \to \mathrm{Hom}(A, W) \to$$
$$\mathrm{Bext}(A, U) \xrightarrow{\alpha_*} \mathrm{Bext}(A, V) \xrightarrow{\beta_*} \mathrm{Bext}(A, W) \qquad (22)$$

and

$$0 \to \mathrm{Hom}(W, A) \to \mathrm{Hom}(V, A) \to \mathrm{Hom}(U, A) \to$$
$$\mathrm{Bext}(W, A) \xrightarrow{\beta^*} \mathrm{Bext}(V, A) \xrightarrow{\alpha^*} \mathrm{Bext}(U, A)\ , \qquad (23)$$

where the product of two successive homomorphisms is always zero. Since balanced subgroups enjoy the transitive property (3.15 *(v)*), a result of Butler and Horrocks [1] shows that (22) and (23) are exact.

We now characterise a number of classes of groups defined by Bext in the same way that Ext defines the class of cotorsion groups. Recall that a group A is cotorsion exactly when $\mathrm{Ext}(J, A) = 0$ for all torsion free groups J. The algebraically compact groups are characterised in like manner; a group A is algebraically compact if and only if $\mathrm{Pext}(X, A) = 0$ for all groups X. With these examples in mind we make the following definition.

8.2 DEFINITION. Let X be a class of groups. Then we denote by X_R the class of all groups A such that $\mathrm{Bext}(A, X) = 0$ for all X in X, and by X_L the class of groups A for which $\mathrm{Bext}(X, A) = 0$ for all X in X. □

Our task is to characterise X_R and X_L for various choices of X .

(A) $X = Ab$, *the class of all Abelian groups.*

The results of Chapters 4 and 6 show that Ab_L is the class of algebraically compact groups, and Ab_R is the class $\overline{A}$.

(B) $X = F$, *the class of all torsion free groups.*

We consider F_L first. If $A \in F_L$ and

$$0 \to U \to V \to W \to 0$$

is an arbitrary balanced exact sequence of torsion free groups, then $\mathrm{Bext}(W, A) = 0$ together with exactness of (24) imply that $\mathrm{Hom}(V, A) \to \mathrm{Hom}(U, A)$ is epic. It follows from (4.4) that A is cotorsion. Conversely, if A is cotorsion then

$$0 = \mathrm{Ext}(W, A) \geq \mathrm{Bext}(W, A)$$

for all W in F and we conclude that F_L is the class of cotorsion groups.

We turn now to F_R . Fuchs has observed that every pure extension of a torsion free group by a torsion group splits. To see this, consider $\mathrm{Ext}(T, J)$ where T is torsion and J is torsion free. By Theorem 52.3 of Fuchs [1], $\mathrm{Ext}(T, J)$ is reduced algebraically compact. As $\mathrm{Pext}(J, T)$ is the first Ulm subgroup of $\mathrm{Ext}(T, J)$ and the first Ulm subgroup of a reduced algebraically compact group vanishes (see Theorem 53.3 and p. 161, respectively, of Fuchs [1]), it follows that $\mathrm{Pext}(T, J) = 0$ as required.

Since $\mathrm{Bext}(C, A) \leq \mathrm{Pext}(C, A)$ for all groups A and C , we see that F_R contains every torsion group. We can say much more:

8.3 THEOREM. *The following are equivalent for a group* A :

(i) $A \in F_R$;

(ii) $A/T(A)$ *is completely decomposable;*

(iii) $A = G/T$ *where* $G \in A^{\Sigma}$ *and* $T \leq T(G)$.

Proof. The proof is cyclic. Suppose $A \in \mathcal{F}_R$. If $0 \to H \to G \to A \to 0$ is a balanced projective resolution of A (see (6.3)) such that $G \in \mathcal{A}^{\Sigma}$ then (3.11) *(c)* implies that

$$0 \to H/T(H) \to G/T(H) \to A \to 0 \tag{25}$$

is also balanced. By assumption, (25) splits and A is a summand of $G/T(H)$. Since $\big(G/T(H)\big)/T\big(G/T(H)\big)$ is completely decomposable, so is $A/T(A)$.

Suppose $A/T(A)$ is completely decomposable. We first show that if A has torsion free rank 1 then $A \in \mathcal{F}_R$. Let a in A have infinite order and set $A/\langle a\rangle = T^*$. Taking a totally projective group H^* and balanced epimorphism $\eta : H^* \to T^*$ we construct the pullback

$$\begin{tikzcd}
0 \arrow[r] & \langle a\rangle \arrow[r] \arrow[d, equal] & G \arrow[r] \arrow[d, "\alpha"] & H^* \arrow[r] \arrow[d, "\eta"] & 0 \\
0 \arrow[r] & \langle a\rangle \arrow[r] & A \arrow[r] & T^* \arrow[r] & 0 .
\end{tikzcd}$$

Since $G \in \mathcal{A}$ and the kernel of α is torsion, A satisfies *(ii)*. For a general group A with $A/T(A)$ completely decomposable, let η be the natural homomorphism $A \to A/T(A)$ and write $A/T(A) = \oplus R_i$ where each R_i has rank 1 . Then $A = \sum \eta^{-1}R_i$. Choosing G_i in $\mathcal{A}$ and epimorphisms $\alpha_i : G_i \to \eta^{-1}R_i$ as above, it is clear that the epimorphism

$$\oplus\, \alpha_i : \oplus\, G_i \to \sum \eta^{-1}R_i = A$$

has torsion kernel.

Finally let A have the form G/T , where $T \le T(G)$ and $G \in \mathcal{A}^{\Sigma}$. Consider the diagram

$$\begin{tikzcd}
 & & & G \arrow[dl, "\theta"'] \arrow[d, "\eta"] & \\
0 \arrow[r] & X \arrow[r] & Y \arrow[r, "\alpha"'] & A \arrow[r] & 0
\end{tikzcd}$$

with balanced exact row, where X is torsion free and η is the natural epimorphism $G \to G/T = A$. Since G is balanced projective there is a $\theta : G \to Y$ with $\eta = \alpha\theta$. Now $\alpha\theta T = 0$ so $\theta T \le \ker \alpha$. Since $\ker \alpha$ is torsion free and θT is torsion, we must have $\theta T = 0$. Hence $\ker \theta = \ker \eta = T$ so there is a homomorphism $\gamma : A \to Y$ with $1_A = \alpha\gamma$. Thus A is a summand of Y. □

We remark that a similar argument shows $\mathrm{Pext}(A, X) = 0$ for all torsion free X if and only if A is the direct sum of a torsion group and a free group, so that F_R is a larger class.

8.4 COROLLARY. *Let A be a direct summand of a direct sum of groups with torsion free rank 1. Then $A \in F_R$.*

Proof. Each such A has $A/T(A)$ completely decomposable. □

8.5 PROPOSITION. *If A has a decomposition basis then $A \in F_R$.*

Proof. Let the decomposition basis be $\{x_i : i \in I\}$, and let $A_i \leq A$ be the unique minimal pure subgroup containing x_i and $T(A)$ for each i. For any a in A we have $na = \sum_{i=1}^{k} n_i x_i$ for integers n, n_i and k. Since $\{x_i\}$ is a decomposition basis, $n | n_i$ for $i = 1, \ldots, k$ so that $a - \sum \frac{n_i}{n} x_i \in T(A)$. Now $T(A) \leq A_i$ for each i and we have $A = \sum_{i \in I} A_i$. Arguing as in (8.3) shows that $A = G/T$ where $G \in A^{\Sigma}$ and $T \leq T(G)$. □

We remark that the example of Rotman and Yen [1] of a group with torsion free rank 2 in $\overline{A}$ but not in A^{Σ} was constructed by factoring out a torsion subgroup from a direct sum of two groups in A. By (8.3) every such example can be obtained this way.

It is easy to see from the remark following (8.3) that a group G satisfies $\mathrm{Pext}(G, X) = 0$ for all torsion free groups X if and only if G

is a pure extension

$$0 \to A \to G \to T \to 0 \tag{26}$$

where A is pure projective and T is torsion - in fact, we simply choose A to be free such that $G = A \oplus T$. We therefore consider what happens when (26) is an arbitrary balanced exact sequence with A balanced projective and T torsion. Let X be torsion free. Applying $\mathrm{Bext}(-, X)$ to (26) gives

$$\ldots\ 0 = \mathrm{Bext}(T, X) \to \mathrm{Bext}(G, X) \to \mathrm{Bext}(A, X) = 0$$

since we have already seen that $\mathrm{Bext}(T, X) = 0 = \mathrm{Bext}(A, X)$. Thus $\mathrm{Bext}(G, X) = 0$ and $G \in F_R$. Conversely, we might ask whether every group G in F_R can be found as a balanced extension (26) with A balanced projective and T torsion. This is equivalent to asking whether every group G with $G/T(G)$ completely decomposable contains a balanced subgroup A which is balanced projective and contains a maximal free subgroup of G. We answer this in the negative with an example.

8.6 EXAMPLE. Honda [1] constructed a p-group G such that $l_p(G) = \Omega$ and $f_\sigma^{\,p}(G) = 1$ for all σ less than Ω. By (2.2) there is an isotype Ω-dense subgroup T of G such that $G/T \cong Z(p^\infty)$. Theorem 2.9 of Nunke [3] shows that the extension

$$\mathrm{e} : 0 \to T \to G \to Z(p^\infty) \to 0$$

is an element of $p^\Omega \mathrm{Ext}\big(Z(p^\infty), T\big)$. Since e clearly does not split, we have $p^\Omega \mathrm{Ext}\big(Z(p^\infty), T\big) \neq 0$. This, together with condition *(ii)* on p. 54 of Chapter 5 shows that there is an element x in $\mathrm{Ext}\big(Z(p^\infty), T\big)$ with infinite order and $U_p(x) = (\Omega, \Omega+1, \Omega+2, \ldots)$. Let M be the unique minimal pure subgroup of $\mathrm{Ext}\big(Z(p^\infty), T\big)$ containing x and T and suppose there is a balanced subgroup A of M such that A has torsion free rank 1 and $A \in \mathsf{A}$. Now A contains an element a of infinite order with $\mathsf{H}_M(x) = \mathsf{H}_A(a)$ and $l_p(A_p) = \Omega$ while it follows from (5.6) that $f_\sigma^{\,p}(A_p)$ is an admissible function.

A glance at the definition of an admissible function shows that $f_\sigma^p(A_p)$ is uncountable for some $\sigma < \Omega$. However A_p is isotype in A and therefore in T so that $f_\sigma^p(A_p) \le f_\sigma^p(T) \le f_\sigma^p(G) = 1$ for all $\sigma < \Omega$, a contradiction. We conclude that no such A exists. □

(C) $X = T$, *the class of all torsion groups.*

We begin with T_L. From what has been said in (B) it is clear that T_L includes the class of all torsion free groups. In fact, the groups in T_L can be characterised *via* their torsion part alone.

8.7 THEOREM. *The following are equivalent for a group* A :

(i) $\mathrm{Bext}(T, A) = 0$ *for all torsion groups* T ;

(ii) $\mathrm{Pext}(T, A) = 0$ *for all torsion groups* T ; *and*

(iii) $T(A)$ *is the direct sum of a torsion complete group and a divisible group.*

Proof. We make some observations from which the proof follows trivially. The pure exact sequence

$$0 \to T(A) \to A \to A/T(A) \to 0$$

induces the exact sequence

$$0 = \mathrm{Hom}\big(T, A/T(A)\big) \to \mathrm{Pext}\big(T, T(A)\big) \to \mathrm{Pext}(T, A) \to \mathrm{Pext}\big(T, A/T(A)\big) = 0$$

for every T in T. Note that $\mathrm{Hom}\big(T, A/T(A)\big)$ is zero since $A/T(A)$ is torsion free and every homomorphic image of a torsion group is torsion, while $\mathrm{Pext}\big(T, A/T(A)\big) = 0$ follows from our discussion on p. 103. If T has no element of infinite p-height for any prime p then (8.1) shows that $\mathrm{Pext}\big(T, T(A)\big) \cong \mathrm{Pext}(T, A) = \mathrm{Bext}(T, A) \cong \mathrm{Bext}\big(T, T(A)\big)$.

Griffith [2] has shown that if G is torsion then $\mathrm{Pext}(T, G) = 0 = \mathrm{Bext}(T, G)$ for all torsion complete groups T if and only if G is itself the direct sum of a torsion complete group and a divisible group. The theorem follows immediately on observing that torsion complete groups have no elements of infinite p-height for any prime p. □

As yet I have been unable to characterise groups in T_R. Clearly, the torsion groups in T_R are exactly the totally projectives, and T_R contains all the balanced projectives. In answer to a long standing question of Baer, Griffith [1] has shown that $\mathrm{Ext}(G, T) = 0$ for all torsion groups T if and only if G is free. He also proved the following closely related result.

8.8 THEOREM (Griffith [1]). *If G is a torsion free group such that $\mathrm{Ext}(G, T)$ is torsion for every torsion group T then G is free.* □

Similarly one might ask whether the torsion free groups in T_R are exactly the completely decomposables. Although we give no answer to this question, it provides motivation for a study of groups which can be represented as extensions in $\mathrm{Bext}(J, T)$, where J is torsion free and T is torsion. These are just the mixed groups having balanced torsion part.

8.9 DEFINITION. We say that A is *lifting* if $T(A)$ is balanced in A. □

Since direct summands are balanced, lifting groups are in some sense a generalisation of splitting mixed groups. Examples of groups whose torsion parts are not balanced abound - for instance if A is reduced and $A/T(A)$ is divisible then $T(A)$ cannot be balanced.

It follows from (2.1) that the torsion part of a mixed group is isotype. As $A/T(A)$ is torsion free and the height of every element in a torsion free group determines its height matrix, we have:

8.10 LEMMA. *A group A is lifting if and only if $T(A)$ is H-nice in A.* □

The following characterisation of lifting groups will be useful.

8.11 LEMMA. *A group A is lifting if and only if every subgroup containing $T(A)$ and having torsion free rank 1 splits.*

Proof. Let A be lifting and let B be a subgroup with torsion free rank 1 such that $T(A) \leq B \leq A$. By (3.15) *(ii)* we have $T(B) = T(A)$ balanced in B. As $B/T(B)$ is balanced projective, B splits.

Conversely, suppose every such subgroup B of A splits. For each x in $A/T(A)$ let B_x be the unique minimal pure subgroup containing $T(A)$ and x. Since B_x splits there is b in B_x with $b + T(B_x) = x$ and $H_{B_x}(b) = H_{A/T(A)}(x)$. It follows from (2.1) that B_x is isotype in A, so $H_A(b) = H_{A/T(A)}(x)$ while (8.10) implies $T(A)$ is balanced in A. □

The next lemma details the restrictions imposed on the height matrix of each element in a lifting group.

8.12 LEMMA. *Let A be a lifting group. Then A is gap-finite and the height matrix of every element of infinite order in A has all entries integers or ∞.*

Proof. The proof of (8.11) shows that such an element can be imbedded in an isotype subgroup of A which splits. The conditions of the lemma are exactly those imposed on the height matrix of every element in a splitting group. □

8.13 PROPOSITION. *The following are equivalent for a group A with torsion free rank 1 :*

(a) A is lifting;

(b) A splits; and

(c) A is gap-finite, and for each prime p the p-height of each element of infinite order in A is an integer or ∞.

Proof. We have already seen that *(a)* and *(b)* are equivalent and that *(b)* implies *(c)*. Megibben [1] has shown that *(c)* implies *(b)*. □

Our next task is to relate a class of groups introduced by C.L. Walker to the lifting groups. Two groups A and B are *quasi-isomorphic* if for some integers m and n there are subgroups G and H of A and B respectively, with $mA \leq G \leq A$ and $nB \leq H \leq B$ and $G \cong H$. C.L. Walker [1] has investigated groups which are quasi-isomorphic to splitting groups - such groups are called *quasi-splitting* by Fuchs [2]. Thus, A is quasi-splitting if there is a splitting subgroup B of A and a positive integer n such that $nA \leq B \leq A$. It is readily seen that B may be assumed to contain $T(A)$. The properties of quasi-splitting groups relevant to our work are contained in:

8.14 THEOREM (C.L. Walker [1]). *Let J be torsion free, T torsion. Then the quasi-splitting extensions of T by J are exactly the elements of finite order in* $\mathrm{Ext}(J, T)$. *If J is countable then every quasi-splitting extension of J by a torsion group splits.* □

8.15 PROPOSITION. *Quasi-splitting groups are lifting.*

Proof. Let A be quasi-splitting, B a subgroup of torsion free rank 1 containing $T(A)$ and C a splitting subgroup of A such that $T(A) = T(C)$ and $nA \leq C \leq A$. Now

$$nB \leq B \cap C \leq B$$

while $B \cap C$ splits, so B itself is quasi-splitting. As $B/T(B)$ is countable (8.14) implies B splits and (8.11) then shows A is lifting. □

We give an example of a lifting group which is not quasi-splitting.

8.16 EXAMPLE. Let J be a homogeneous, indecomposable torsion free group of type $(0, \ldots, 0, \ldots)$ having finite rank ≥ 2. The existence of such groups is proved in Fuchs [2]. By (8.8) there is a torsion group and an extension

$$e : 0 \to T \to G \to J \to 0$$

such that e has infinite order in $\mathrm{Ext}(J, T)$. It follows from (4.1) that $e \in \mathrm{Bext}(J, T)$, while (8.14) shows that G cannot be quasi-splitting. $\square$

We have shown that the torsion parts of $\mathrm{Ext}(J, T)$ and $\mathrm{Bext}(J, T)$ coincide for every torsion free group J and torsion group T . From (4.1), we have $\mathrm{Bext}(J, T) = \mathrm{Ext}(J, T)$ for all torsion T whenever J is homogeneous of type $(0, \ldots, 0, \ldots)$. As a partial converse we have:

8.17 LEMMA. *Let* J *be a torsion free group containing elements of type greater than* $(0, \ldots, 0, \ldots)$. *Then there exists a torsion group* T *such that* $\mathrm{Bext}(J, T)$ *is a proper subgroup of* $\mathrm{Ext}(J, T)$.

Proof. Let x in J have type greater than $(0, \ldots, 0, \ldots)$. Now $\langle x \rangle_*$ is not free so $\mathrm{Ext}(\langle x \rangle_*, T) \neq 0$ for some torsion group T . Suppose $0 \neq e \in \mathrm{Ext}(\langle x \rangle_*, T)$. The natural injection $i : \langle x \rangle_* \to J$ induces the epimorphism $i^* : \mathrm{Ext}(J, T) \to \mathrm{Ext}(\langle x \rangle_*, T)$. If $f \in \mathrm{Ext}(J, T)$ and satisfies $i^* f = e$ then $f \neq 0$. It follows from (8.11) that $f \notin \mathrm{Bext}(J, T)$. $\square$

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